Having introduced the friction the previous class, we worked through a number of examples. Remember that this is the study of Newton’s Laws, with friction simply being one interesting force.

**Second Newton’s Law.** Written out in components (implying a specific coordinate system):

\[
\sum_i F_i = m \ddot{a} \iff \sum_i F_{ix} = ma_x \quad \sum_i F_{iy} = ma_y
\]

Components are given by definition as: \( F_x = F \cos \theta \) and \( F_y = F \sin \theta \). The angle “\( \theta \)” runs from the positive \( x \) axis, toward the vector, counter-clock wise. This guarantees correct value and sign.

Note the one possible snag with friction: if the object under study isn’t moving – but is held in place instead, at least partly due to friction – one need be careful with figuring out which way the friction acts.\(^1\) Other than this, when there is motion, it is simple: the friction acts opposite to it, so it is represented by a force directed opposite to where the object is moving. With that decided, you approach problems exactly the same way as so far: set up all forces, and write down Newton’s Laws, in your coordinate system. Then you can work the equations toward whatever is asked. Some (simple) decisions based on physical reasoning are often necessary, and/or helpful.

We worked through nearly all typical setups. Most of the time, you’ll need to find an expression for acceleration, even if this may not be the final answer. Very often, this involves the following.

Use the definition of friction, \( F_{fr} = \mu F_N \), to eliminate \( F_{fr} \) from the equation.\(^2\) This brings in \( F_N \) instead, and most of the time you’ll then need to eliminate that too: use the other equation, with \( F_N \), to ‘solve’ for it (obtain an algebraic expression for it), and substitute this instead of \( F_N \) in the first one. That one remaining equation then allows you to solve for acceleration. Note that this is only one typical example. In various problems, you may be given some kinematical data (times, speeds, distances ... at various points), which allows you to actually find the acceleration using them. Then you may be asked for other things, say for some unknown force or \( \mu \). But you may still use your solution (formula) for acceleration, except that now you would turn it around to solve for what is asked. Regardless of what the problem actually asks (and it can be all kinds of things), most of the time finding an expression for acceleration allows you to then find that.

We first reviewed the example of the last class, of what force is needed to keep something still, pressed against the wall. Then we looked at the typical example of finding the ‘stopping distance.’ (Recall the kinematics: there the acceleration was typically given. Now we know how to find it.) Next we looked at an object on a flat surface, pushed under an angle. Then we went back to pushing something against a wall, but now with a force acting under an angle too. This brought back the discussion of static friction and its direction: we may be pushing so little as to barely stop the thing from sliding down the wall, in which case the friction must be directed upward (since without it the object would clearly slide down); or, we may be pushing so hard that the object almost accelerates upward, in which case the friction fights that, and is so directed downward.

Finally we discussed a very typical example of an object on an inclined plane. Note that we have already worked through this in the past, only without specifically naming the force of friction.

In most of these examples, we solved for acceleration. This is in general a solution to our dynamical problem, since it is then possible to work out the details of the motion. But, in problems, one can be asked for all kinds of other things. You would still normally want to solve for acceleration, and then use that expression to finish the problem.

---

\(^1\)Check which way the object would move in such a setup, but without friction. The friction acts the other way.

\(^2\)Typically the \( x \) equation. It is often convenient to direct \( x \) along the motion, and friction is that way too.
Today we worked through examples. First we reviewed the last class's example—please see those notes.

A typical problem: stopping distance, from a given speed ("\(v_0\)"), knowing the coefficient of friction ("\(\mu\)").

\[
\begin{align*}
\vec{F}_{fr} &= \mu F_N \\
- (\mu F_N) &= \text{max} \\
F_N - mg &= 0 \\
\Rightarrow F_N &= mg, \text{ and:} \\
- \mu F_N &= \text{max}, \quad a_x = -\mu g. \quad (\text{very simple!})
\end{align*}
\]

Now we can solve the kinematical question.

Use: \(v_f^2 = v_0^2 + 2a_x \Delta x\), since \(v_f = 0\)

So, \(0 = v_0^2 + 2a_x \Delta x\), for this question.

And with \(a_x = -\mu g\) we get:

\[
\begin{align*}
\Delta x &= -\frac{v_0^2}{2a_x} = -\frac{v_0^2}{2(-\mu g)} \\
\Delta x &= \frac{v_0^2}{2\mu g}
\end{align*}
\]

Note how many other questions can be answered with this. (For example: a car driving at 30 mph, slams the brake, and stops in 12 meters. What (a driver does!) is the effective coefficient of friction?)
Push down on something, with a given force, under a given angle – what is the acceleration?

\[ F_0 \cos \theta - F_{fr} = ma_x \]

\[ F_0 \sin \theta - F_{gr} + F_N = ma_y = 0 \] (doesn't go up/down, slides along)

\[ F_{fr} = \mu_k F_N \]

Note: for the purpose of working through this example, we simply take it that this does accelerate, strictly we would need to first check whether static friction is defeated.

Solve the above equations. Using \( F_{fr} = \mu_k F_N \):

\[ F_0 \cos \theta - \mu_k F_N = ma_x \]

and from y-equation:

\[ F_N = mg + F_0 \sin \theta \], which eliminates \( F_N \):

\[ \Rightarrow F_0 \cos \theta - \mu_k (mg + F_0 \sin \theta) = ma_x \]

So:

\[ a_x = \frac{1}{m} \left[ F_0 \cos \theta - \mu_k (mg + F_0 \sin \theta) \right] \]

Look at the physics here; we can see exactly what makes it go \( \left( \frac{F_0 \cos \theta}{m} \right) \), and what slows it down \( \left( -\mu_k (\ldots) \right) \), and why: \( -\mu_k (mg + F_0 \sin \theta) \) vertical component of that force, pressing it against pressure of weight on the surface the surface.
Push it against the wall ... but under an angle.
How hard \( F_0 = \) so that it doesn't accelerate at all?

Need to see which way it would go without it:

\[
\Sigma F_y = F_0 \sin \theta - mg.
\]

Now enter numbers: if this is positive, we are pushing hard enough to accelerate it upward (if there were no friction), so the friction acts downward; if the number comes out negative, it would go down, so friction is upward.

For this example take that it would slide down, so friction is upward.

\( x \) \( F_N - F_0 \cos \theta = \max = 0 \)

\( y \) \( F_{fr} + F_0 \sin \theta - F_{gr} = \max = 0 \)

\( F_{fr} \leq \mu s F_N \)

we'll use maximum friction!

Solve these.
Possible motion is

\[ F_N - F_0 \cos \Theta = 0 \] (\( a_y = 0 \); along the wall only)

\[ F_0 + F_0 \sin \Theta - mg = 0 \] (\( a_y = 0 \) required here)

\[ F_{\text{net}} = \mu_s F_N \]

\[ \Rightarrow \mu_s F_N + F_0 \sin \Theta - mg = 0, \quad F_N = \frac{mg - F_0 \sin \Theta}{\mu_s} \]

\[ \Rightarrow \frac{mg}{\mu_s} - F_0 \frac{\sin \Theta}{\mu_s} - F_0 \cos \Theta = 0. \text{ Need } F_0: \]

\[ \frac{mg}{\mu_s} - F_0 \left( \frac{\sin \Theta}{\mu_s} + \cos \Theta \right) = 0, \]

\[ \Rightarrow F_0 = \frac{mg/\mu_s}{\sin \Theta + \mu_s \cos \Theta} = \frac{mg}{\sin \Theta + \mu_s \cos \Theta} \]

So:

\[ F_0 = \frac{mg}{\sin \Theta + \mu_s \cos \Theta} \]

Does this make sense — do you understand the
\((\sin \Theta + \mu_s \cos \Theta)\)? What would a minus sign mean?
(what would be different in the setup?)

What is the best angle
to push under? (so the necessary force is minimal?)
Let's say you push with a lot more force, making sure it goes up. What is acceleration?
Say, we 'shore' something up a ramp, with a known speed, inclination angle, \( \mu \), etc — what is acceleration? (Then we can ask all kinds of other questions!)

\[
\begin{align*}
- F_N - F_{gr} \sin \Theta &= \max \\
F_N - F_{gr} \cos \Theta &= \max (\approx \theta) \\
F_{fr} &= \mu_k F_N.
\end{align*}
\]

Solve the system:

\[
\begin{align*}
- \mu_k F_N - \mu g \sin \Theta &= \max, \quad \text{and} \\
F_N - \mu g \cos \Theta &= 0 \implies F_N &= \mu g \cos \Theta, \\
- \mu_k(\mu g \cos \Theta) - \mu g \sin \Theta &= \max
\end{align*}
\]

\[
\implies a_x = - g \sin \Theta - \mu_k \mu g \cos \Theta.
\]

(What if it were sliding down the plane instead?)

Come up with, and answer, as many questions as you can! How far does it go? Does it stay there? If not, how long does it take it to come back down, and with what speed? What if there is another force there, say horizontal? What is the acceleration?