We worked through an example of objects pushed against each other (Third Newton’s Law).

**Newton’s Laws.** Second law, written out in components, and the Third Law:

\[
\sum_i F_i = m \ddot{a} \quad \iff \quad \sum_i F_{ix} = ma_x \\
\sum_i F_{iy} = ma_y
\]

For objects A and B interacting: \( \vec{F}_{AB} = -\vec{F}_{BA} \)

Components are given by definition as: \( F_x = F \cos \theta \) and \( F_y = F \sin \theta \). The angle “\( \theta \)” runs from the positive \( x \) axis, toward the vector, counter-clock wise. This guarantees correct value and sign.

In the last class we looked in detail at how to handle a ‘pulling’ interaction between two objects. Now we look at pushing: two blocks are pressed against each other, so that there is an interaction between the surfaces of their contact. This will round up the basic statement of how to treat contact forces in Newtonian mechanics, and of the Third Newton’s Law.

So we have two blocks on a flat surface, and one of them has a force pushing on it. Let’s say that this is the block on the left (we’ll call it “A”) and the force is pushing on it from the left, to the right. This force makes the block A go to the right, but only by pushing the other block along. We’ll call the other block “B.” Their surfaces are pressed against each other, interacting in a manner just like we discussed when dealing with the normal force.

This interaction is very handily represented in the following way. Each object is being pushed back, feeling a ‘force’ pushing it away from the other one. So we can model this interaction by introducing a ‘pair of forces,’ such as: one force from the pair acts on one of the blocks, pushing it back from the surface; the other acts on the other, pushing this one back too. These forces are equal in magnitude, and clearly opposite in direction. Given how we labeled the blocks, we can call them: \( F_{BA} \) (short of \( F_{B\rightarrow A}, \) block \( A \) acted upon by \( B \)), and \( F_{AB} \) (\( B \) acted on by \( A \)). This is how the Third Newton’s Law models interactions, and is summarized with: \( \vec{F}_{AB} = -\vec{F}_{BA} \).

Then, we proceed exactly like we did with the ‘pulling interaction’ of the last class. Focus on one object, and identify all forces acting on it, including the appropriate force from the ‘pair’ (one only acting on one object); then write down the 2nd Newton’s Law equations for this object, as if nothing else existed. Then do the same for the other object. The force \( F_{BA} \) is of course in \( A \)’s equations (representing the fact that it is physically affected by \( B \)). Likewise, \( B \)’s equations have \( F_{AB} \), taking care of \( B \)’s physical interaction with \( A \). This way we constructed two sets of equations, for motion of blocks \( A \) and \( B \), which also account for their interaction. And there is the relation between their accelerations, needed to be honored for these systems of equations to be consistent with each other. This is done exactly the same way as discussed in the previous class, but with pushing it is easier: the blocks can only go the same way, so \( a_{1x} = a_{2x} \).

See the “Practice Homework” on our Web site’s Materials page, for discussion of treatment of contact forces, and examples of systems interacting in this way.

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1. This interaction *is* a normal force, in the sense in which we had introduced it. Recall the discussion of an object sitting on a desk (for example), with a ‘normal’ force perpendicular (‘normal’) to the surface of their contact. That was exactly the same as discussed now, except that then we dismissed the question of what happens to the desk.

2. But solve the problems on your own first!
For the motivation for this example, see c19 notes: if we push hard on one end of the object, how is this force distributed through the object? (For example: is it good enough to make the front part of the object very hard—is the force inside smaller and smaller, as we look further from the surface?).

To study this, we can look at two objects, pushed against each other.

\[ \mathbf{F}_{12} = -\mathbf{F}_{21} \]

represent forces acting on center of mass— for each object separately!

Now write down 2nd Newton’s Law, for each:
c21, Fri May 14

Object 1:
\[ F_0 - F_{21} = m_1 a_{1x} \]
\[ F_{n1} - F_{g1} = m_1 a_{1y} = 0 \]

So we get systems of equations:
\[ F_0 - F_{21} = m_1 a_{1x} \]
(remember, \( a_{1x} = a_{2x} = a \))
\[ F_{12} = m_2 a \]
\[ F_{n1} - F_{g1} = 0 \]
(\( F_{n1} \) from \( F_{g1} = \mu_1 F_{n1} \))

Object 2:
\[ F_{12} = m_2 a_{2x} \]
\[ F_{n2} - F_{g2} = m_2 a_{2y} = 0 \]

Quite simply:
\[ F_0 - F_{21} = m_1 a, \text{ and } F_{12} = m_2 a \]

Now use \( F_{12} = F_{21} \), (again: \( +F_{12} = +F_{21} \) — the opposite sign of \( F_{12} \) and \( F_{21} \) has been accounted for.)

is solved. Go for acceleration first...
\[ F_0 - (m_2a) = m_1a \Rightarrow a = \frac{F_0}{m_1 + m_2} \]

This could've been found right away: the blocks move as one, \( F_0 = (m_1 + m_2) \cdot a \) treat two blocks as one object.

Use this in the second equation:

\[ F_{12} = m_2 \left( \frac{F_0}{m_1 + m_2} \right) \Rightarrow F_{12} = F_0 \cdot \frac{m_2}{m_1 + m_2} \]

So the force on the interface (surface between objects) is a fraction of the external force: the 'remaining mass' divided by total mass, we find that the force decreases from the surface into the object (and linearly with mass).

Now one could from this infer:

\[ \text{on this surface: } F_e = F_0 \cdot \frac{L}{L}, \text{ presuming a constant cross-section and density.} \]

This was an example. Studying contact forces goes way beyond this — it was only a motivation.
Set up a more involved example:

**Object 1:** \( m_1 > m_2 \) (why?)

- \( F_{21} - m_1 g \sin \theta + F_{N1} = m_1 a_{1x} \)
- \( F_{N1} - m_1 g \cos \theta = m_1 a_{1y} = 0 \)
  \( \Rightarrow F_{N1} = m_1 g \cos \theta \).

Also, \( F_{fr1} = \mu_1 F_{N1} \)

\(- F_{21} - m_1 g \sin \theta + \mu_1 (m_1 g \cos \theta) = m_1 a_{1x} \)

**Object 2:** \( m_1 < m_2 \) (why?)

\( F_{21} + F_{fr2} - m_2 g \sin \theta = m_2 a_{2x} \)
\( F_{N2} - m_2 g \cos \theta = m_2 a_{2y} = 0 \)
  \( \Rightarrow F_{N2} = m_2 g \cos \theta \),
  and \( F_{fr2} = \mu_2 F_{N2} \).

\( F_{21} + \mu_2 (m_2 g \cos \theta) \)

\(- m_2 g \sin \theta = m_2 a_{2x} \)

Note, \( a_{1x} = a_{2x} = a \).

So, we get:

**Object 1:** 
\(- F_{21} - m_1 g \sin \theta + \mu_1 m_1 g \cos \theta = m_1 a \)

**Object 2:** 
\( F_{21} - m_2 g \sin \theta + \mu_2 m_2 g \cos \theta = m_2 a \)

We can see physics here directly: both are pulled down the plane by gravity (-), up the plane by friction (+) — resisting the motion. Also
block 1 is pushed down the plane by block 2 \((-F_{21})\), while block 2 is pushed up the plane by block 1 \((+F_{12})\).

Now we can make use of \(F_{12}=F_{21}\) to get \(a\).

For example, add the equations:

\[
(-F_{21} - m_1 \cdot g \cdot \sin\theta + M_{k1} \cdot m_1 \cdot g \cdot \cos\theta) + (F_{21} - m_2 \cdot g \cdot \sin\theta + M_{k2} \cdot m_2 \cdot g \cdot \cos\theta)
\]

\[
= (m_1 a) + (m_2 a),
\]

\[
(m_1 + m_2) a = -m_1 g (\sin\theta + M_{k1} \cos\theta) - m_2 g (\sin\theta + M_{k2} \cos\theta)
\]

\[
= - (m_1 + m_2) g \cdot \sin\theta - g (m_1 M_{k1} \cos\theta + m_2 M_{k2} \cos\theta)
\]

\[
\Rightarrow \quad a = -g \sin\theta - \frac{m_1 M_{k1} + m_2 M_{k2}}{m_1 + m_2} \cdot g \cos\theta
\]

"moving like one" part

"sort of effective friction, between their different \(M_{k1}\) and \(M_{k2}\)."

Study the above for different limiting cases:

\(M_{k1} = M_{k2}\); \(m_1 = m_2\); \(m_1 \gg m_2\), or \(m_2 \gg m_1\).

Put this expression back into either equation above (instead of \(a\)) to get \(F_{12}\) or \(F_{21}\).

Can you get a 'nice' expression — see physics in it?