We proceed with momentum conservation. We also worked out simple examples of the use of \( \Delta \vec{p} = \int \vec{F} dt \). (But first we reviewed another example of Newton’s Laws: two blocks pulled upward.)

Consider a system of objects, each carrying some momentum, acted upon by some external forces \( \sum \vec{F} = \vec{F}_{\text{net}} \). These induce the change of total momentum of the system \( \Delta \vec{P}_{\text{tot}} \). If the external forces happen to add to zero, the total momentum is conserved: it stays constant, does not change.

\[
\vec{F}_{\text{net}} = \sum \vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \Delta \vec{P}_{\text{tot}} = \int_{t_i}^{t_f} \vec{F}_{\text{net}} dt.
\]

When \( \vec{F}_{\text{net}} = 0 \) \( \Rightarrow \ \vec{P}_{\text{total}} = \sum \vec{p} = \text{const} \)

This is a vector equation, and when it holds it implies conservation of momentum per degree of freedom: \( \sum p = \text{const} \Rightarrow \sum p_x = \text{const} \) and \( \sum p_y = \text{const} \) (in two dimensions). The above discussion of the 2nd Newton’s Law holds for components separately, too:

\[
\Delta p_x = \int_{t_i}^{t_f} F_{\text{net},x} dt \quad \Rightarrow \quad \Delta p_y = \int_{t_i}^{t_f} F_{\text{net},y} dt
\]

So if a component of the total external force is zero, momentum along that direction is conserved.

For example, if \( F_{\text{net},x} = 0 \) \( \Rightarrow \ \sum p_x = \text{const} \). We looked at examples of some of possibilities: \( F_{\text{ext},x} = 0 \) \( \Rightarrow \ \vec{P}_{\text{tot},x} = \text{const} \); and \( \int F_{\text{ext}} dt = \Delta \vec{P}_{\text{tot}} \neq 0 \) (but while \( \sum F_{\text{ext},y} = 0 \)).

Imagine an object coming at the other, which is at rest. In the collision they stick together, and move on as one. If there are no external forces acting along this line, we can use conservation of momentum, or more precisely its \( x \) component: \( \sum p_{ix} = \sum p_{f,x} \). In words, sum of \( x \) components of ‘initial’ momenta \( (p_{ix}) \) is equal to the sum of \( x \) components of ‘final’ momenta \( (p_{f,x}) \). Note that these ‘initial’ and ‘final’ refer to any two moments really, but in this example we choose them to be (right) ‘before’ and ‘after’ the collision. Let’s call the incoming particle ‘1’ and target ‘2,’ and let us assume that the speed of the incoming particle may be known, as well as masses of both. Then the above equation becomes: \( m_1 v_{1i} = (m_1 + m_2) V_f \). So we found the speed with which the two continue (together): \( V = v_{1i} m_1/(m_1 + m_2) \). This is an example of a perfectly inelastic collision, the term we’ll learn more about when we discuss energy.

In general, if they were both moving, and didn’t stick together (but parted ways after the collision), we do the same: \( \sum p_{ix} = \sum p_{f,x} \). This is: \( m_1 v_{1i,x} + m_2 v_{2i,x} = m_1 v_{1f,x} + m_2 v_{2f,x} \).

Sometimes we have to accept this cumbersome notation, as there are things to keep track of. Imagine again the same collision, with the target particle at rest initially, but with objects now not sticking together afterward. Imagine that again we know the incoming speed, and the masses. Look at the above equation: after the collision we now have two unknowns \( (v_{1f,x} \) and \( v_{2f,x} \)), in this one equation! We cannot solve this, or proceed with it. We need more physics, more equations relating these quantities, to form a system of equations. (The conservation of energy will do it.)

Cases of ‘explosion’ are dealt with in the same way: the initial momentum of a piece is a zero, so the total momentum of fragments flying out must add to zero. Note that even in such a violent and messy event as an explosion, the momentum conservation holds! (And so is very useful, since many other laws of physics become difficult to use.) If the object was at rest before its parts flew out, and if only two parts went out, then the math becomes very easy: \( 0 = -m_1 v_1 + m_2 v_2 \) (since they must go off in the opposite directions, back-to-back). If the exploding object were moving prior to breaking up, then its momentum becomes the total momentum of parts, and the above “0” should be adjusted accordingly. Note that all this is in one dimension. In two dimensions one looks at components (directions) separately, as we will in the next class. Finally: this holds for many other examples, not only actual explosions: two skaters pushing away from each other, etc.

We also looked at a particle bouncing off of a wall, and calculated the average force of that interaction. We discussed both cases when it hits the wall straight, and under an angle. See notes.
Conservation of momentum, and \[ \Delta \vec{P} = \int F \, dt \], uses. But first go through a Newton's Laws example.

\[ F_0 = 10N \]
\[ m_1 = 3kg \]
\[ F_T = \ ? \]
\[ m_2 = 1kg \]

\[ \begin{align*}
F_0 - F_T - m_1 g &= m_1 a_1 y \\
F_T - m_2 g &= m_2 a_2 y
\end{align*} \]

In this case, clearly \( a_1 y = a_2 y = a y \) (call it that).

Solve the system for \( F_T \).

\[ F_0 - F_T - m_1 g = m_1 a_1 y \]
\[ F_T - m_2 g = m_2 a_2 y \]
\[ a_1 y = \frac{F_T - m_2 g}{m_2} \]
\[ a_1 y = \frac{F_T - m_1 g}{m_1} \]
\[ F_0 - F_T - m_1 g = m_1 \left( \frac{F_T - m_2 g}{m_2} \right) = \frac{m_1}{m_2} (F_T - m_1 g) \]
\[ F_0 = F_T + \frac{m_1}{m_2} F_T \]
\[ F_0 = \left( \frac{m_2}{m_1} + \frac{m_1}{m_1} \right) F_T = \frac{m_2 + m_1}{m_1} F_T \]
\[ F_T = \frac{m_2}{m_1 + m_2} F_0 = \frac{m_2}{m_1 + m_2} 10N = \frac{4}{5} \cdot 10N = 2.5N \] (no calculator!)

Numbers: \( F_T = \frac{1kg}{1kg + 3kg} \cdot 10N = \frac{1}{4} \cdot 10N = 2.5N \)

Recall how the force distributes through an object:

\[ F(x) = \frac{x}{L} F_0 \] for constant cross-section, density.

We get the same result:

\[ F = \frac{m_2}{(m_1 + m_2)} F_0 \]
Review last class:

\[ \Sigma F = ma = m \frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} \]

\[ \Sigma \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}}{dt} \]

\( m = \text{const} \)

Example:

\[ \mathbf{v}_i \rightarrow +x \]

\[ m_1 \rightarrow v_i \]

\[ m_2 = 0 \]

(stick) \[ \rightarrow ? \]

\[ V = ? \]

In this case:

\[ \Sigma p_{ix} = \Sigma p_{fx} \rightarrow m_1 v_i = (m_1 + m_2) V \]

\[ \Rightarrow V = \frac{m_1}{m_1 + m_2} v_i \quad \text{done!} \]

Look at the opposite; in a sense; "explosion."

An object explodes, and pieces fly out; say that it is two pieces.

First; if it is exactly two pieces, they must go along a line.

Otherwise:

So, they fly off back to back.
So, we have:

\[ \Sigma \vec{p}_i = \Sigma \vec{p}_f, \text{ or } \Sigma p_{ix} = \Sigma p_{fx}, \]

\[ \Sigma p_{iy} = \Sigma p_{fy} \]

In this example, \( \vec{p}_i = 0 \).

Then:

\[ 0 = P_{1f,x} + P_{2f,x} \rightarrow \text{but since they initial momentum must fly in opposite directions, we may well say:} \]

\[ \begin{align*}
0 &= -m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \\
 0 &= -m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} \\
\end{align*} \]

So

\[ O = -m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \Rightarrow m_1 \vec{v}_{1f} = m_2 \vec{v}_{2f} \]

This makes perfect sense: the heavier part carries away less speed. (That's what "momentum" is about, as quantity: \( m \) and \( v \).)

---

What if the object were moving? \( \rightarrow \) (all on one line)

\[ M \rightarrow V \]

\[ M \rightarrow V \]

For now, take the case when they go off along the same line.

But take the example when a part simply separated:

\[ M \rightarrow V \]

\[ m_1 \rightarrow V \]

\[ m_2 \rightarrow V \]

right after it splits off, this part still goes at the same speed! The momentum of the system:

\[ MV = m_1 \vec{V}_{1f} + m_2 \vec{V}_{2f} \]

\[ m_1 \vec{V}_{1i} + m_2 \vec{V}_{2i} \]

in principle, 2 unknowns!
Note the input 'from physics' - we 'reason' (so to say) that the separated part has the same speed. Then, \[ Mv = m_1 v_{1f} + m_2 v \Rightarrow v_{1f} = v \text{ (since } m_1 + m_2 = M \). The speed doesn't change!

Let's shift gears; throw a ball at the wall. If it bounces off nicely, what is (an estimate for) the forces that acted during that brief collision?

\[ \text{magnitude of} \quad \Delta P_x = \int F_x \, dt. \quad \text{Also, need to 'assume' that the intense, varying force is 'nice' enough (symmetric), so that the integral can be approximated as:} \int F_x \, dt \approx \overline{F_x} \cdot \Delta t \]

Finally, we also have to estimate the duration of this collision: let's say that it takes between 1ms and 10ms.

So: \[ \Delta P_x = \int F_x \, dt, \quad \Delta P_x \approx \overline{F_x} \cdot \Delta t \]

"\( \Delta P_x \)" means: \[ \Delta P_x = P_{f,x} - P_{i,x} \]. In the above coordinate system:

\[ P_{f,x} = -m_1 v_{1f} \]
\[ P_{i,x} = +m_1 v_i \]
\[ \int \Delta P_x = \overline{F_x} \Delta t, \]
\[ -m_1 v_{1f} - (+m_1 v_i) = \overline{F_x} \Delta t \]
\[ \Rightarrow \overline{F_x} = - \frac{m_1 (v_{1f} + v_i)}{\Delta t} \]
\[ F_x = - \frac{m (v_f + v_i)}{\Delta t} \]

(minus sign: force on the ball acts to the left, indeed.

(Note that the wall suffers the exact same force - acting on it to the right.)

Note: the speeds add - the (overall) change of momentum in this case means that the ball is stopped ... and sent back!

Numbers? Well, it depends on "M" and "V's" - but the time of a collision is surely short (1-10ms), so the forces are rather large.

If the bounce is 'perfect' - the ball comes back at the same speed: \[ F_x = - \frac{2mv_v}{\Delta t} \]. Simple.

---

How about having an angle about it?

\[ \Delta p = \int F \, dt \iff \Delta p_x = \int F_x \, dt, \quad \Delta p_y = \int F_y \, dt. \]

So in \( x,y \) we have:

\[
\begin{align*}
-m v_f x - (m v_i x) &= \int F_x \, dt \\
m v_f y - m v_i y &= \int F_y \, dt
\end{align*}
\]

In a 'perfect' bounce, \( F_y = 0 \)

and \( v_f y = v_i y \). The ball keeps going up the same, only reflects, (along \( y \)) (in \( x \))

forces act perpendicular to the surface!