We started talking about two dimensions, and introduced mathematical tools for describing them. We also started with 2-dimensional kinematics. We continue with both these topics on Wednesday.

2 dimensions. For motivation we looked at how to describe a position in two dimensions; and how to (fully) describe velocity in two dimensions. (The word ‘speed’ I used so far is a little colloquial, and can only be used for velocity’s magnitude.) The position description is intuitive: in two dimensions we clearly need two coordinates, and as most know, position is a “vector.” It turns out, a vector is any quantity that needs multiple components (“coordinates”) for its complete description.\(^1\) Giving all components then completely specifies a vector. Often, one simply lists them all in order, in whatever coordinate system is being used: vector \(v = (v_1, v_2, ..., v_N)\). This would be an \(N\)–dimensional vector, as it has \(N\) components. We will only need 2 dimensions.

In 2 (and 3) dimensions, a vector is often labeled by a little arrow over its symbol. So, a 2–dimensional “vector \(A\)” would be called \(\vec{A}\). The components (coordinates) of a vector themselves do not bare any arrows above them – those are just numbers. (In this class; in general, vectors can be made up of all kinds of objects as their components, not necessarily simple numbers.)

\[
|\vec{A}| = \sqrt{A_x^2 + A_y^2}
\]

Specify a vector via its coordinates, either way:

\[
\vec{A} = (A_x, A_y) \quad \text{ordered pair of coordinates}
\]

\[
\vec{A} = A_x \hat{i} + A_y \hat{j} \quad \text{using “unit vectors”}
\]

For example, the position: \(\vec{r} = (x, y) = x \hat{i} + y \hat{j}\).

Note: magnitude (“length”) of a vector is often labeled by its symbol alone (no ‘arrow’): \(|\vec{A}| \equiv A\).

For example, the vector \(\vec{A}\) above has coordinates \((10, 5)\): 10 units (of length) along \(x\) axis, and 5 units in \(y\). So we can write it either as \(\vec{A} = (10, 5)\) or \(\vec{A} = 10 \hat{i} + 5 \hat{j}\). Its magnitude is:

\[
|\vec{A}| = \sqrt{A_x^2 + A_y^2} = \sqrt{10^2 + 5^2} = 11.2.
\]

(This example vector does not have physical dimensions, so there are no units.) Or, if the magnitude (“length”) and the angle are given (\(\theta = 30^\circ\) above), the components (coordinates) can be calculated as: \(A_x = A \cos(\theta) = 11.2 \cos(30^\circ) = 9.70\). (The same way for \(A_y\), with the sine function instead.) Note the loss of precision: I had rounded the number for the magnitude (the 11.2 above), and now when I use this I lose some precision.

All this applies directly to velocity. Given a velocity \(\vec{v}\), its components are \(v_x = |\vec{v}| \cos(\theta)\), and \(v_y = |\vec{v}| \sin(\theta)\). The velocity components have a different, less intuitive, meaning than those of position: they are ‘speeds’ with which the object is covering distance along each coordinate direction. So the \(v_x\) component is the speed of motion along the \(x\) direction; in our tsunami–ridden class example, it is the speed with which we are actually getting away from the ocean.

Basics of vectors are covered at great length in the Chapter 3 of the book. Much of that you know, some of it you don’t: please read as needed. In principle, one should know nearly all of it. For the time being, we need very little more than shown above. See class notes, for this class and the next.

Vectors are everywhere in science. There is a lot more to them than the above basic rules for their manipulation. We will often talk about the vector nature of quantities we work with.

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\(^1\)For something to be a vector, it also has to satisfy certain transformation properties. This is beyond our class.

\(^2\)Note that something that is fully described by one component (say, a number) is not a vector, but a scalar.
Kinematics in 2 dimensions. We introduced the very first problem dealing with motion in 2 dimensions: push something away, say out of the window, strictly horizontally. How do we describe its motion? For example, where does it land? (Note this week’s lab!)

As it turns out, this is just an example of a very deep and far reaching principle in physics. The motion of that object can be studied in each of (the two) dimensions – separately from each other. We say that there are two degrees of freedom, and they can be treated independently. This is an extremely powerful principle. Problems in physics that do not lend themselves to such “decomposition” are very difficult, and require complex approaches and techniques.

Concretely this means that we can use our kinematics equations of motion for the \(x\) direction, and separately from that, we may write another set of equations for motion in the \(y\) direction. It is as if the object moves along \(x\), and along \(y\), independently! Of course, it doesn’t: it goes on a curved path (parabola), not along \(x\) and \(y\)!

The object is pushed horizontally (with respect to the ground level), with the initial speed of \(v_0\). As for what affects its motion, we will omit an account of air resistance, and so the only force acting on it is the force of gravity, pulling straight down, (strictly) vertically with respect to the ground. Now, we take the leap: this provides the acceleration of \(9.8 \, \text{m/s}^2 \equiv g\). (We have already been doing this for the free-fall motion in 1 dimension.) This fact is obtained easily using Newton’s Laws, and we will do so in a week’s time. For now, we simply use it. To summarize: the initial speed is horizontal (given in the symbolic form, as \(v_0\)); the only force acting (in our approximation) is gravity, providing acceleration of \(g\) (downward).

We set our coordinate system as follows: the \(x\) axis runs horizontally, positive away from us; and the \(y\) axis runs vertically, positive upward. (Since in this case we know what happens – when things are pushed out of the window – we can expect that: in our coordinate system the object starts moving along the positive \(x\) axis, and will be moving along the negative \(y\) axis.)

Now we can go back to carrying out our program, of setting up equations of motion independently for \(x\) and \(y\) components. The kinematics equation for the \(x\) coordinate in general is:

\[
x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2;
\]

and since there is no acceleration along that direction, the “\(x\)-motion” (how distance is covered in the horizontal sense) is described by:

\[
x = v_{0x}t.
\]

Finally, since we pushed it out strictly horizontally, all of the initial speed is along the \(x\) direction, and \(v_{0x} = v_0\). So:

\[
x = v_0t.
\]

This is how distance is covered horizontally as the object falls. (For example, it tells us how far from the wall the object is at any point in time.)

We can also apply the general kinematics equation for velocity:

\[
v_x = v_{0x} + a_xt
\]

becomes \(v_x = v_{0x} = v_0\). We now have a full description of the “\(x\) component” of the motion.

Similarly, parsing the conditions of our problem, we see that the \(y\) coordinate is described by:

\[
y = -(1/2) gt^2.
\]

The \(y\) component of the velocity is:

\[
v_y = -gt.
\]

Thus we got the full equations of motion for that thing we pushed out of the window. (In our approach, we left out the air resistance. Remember also that we are modelling our real–world object as a perfect material point – using the ‘particle model.’)

But, after all, this is one object, moving along its real path: these equations have to (somehow) be related to each other. They are connected indeed, via time: time is the same in both sets of equations. Other than being necessary, this also allows us to use these equations together – which is how we solve problems. (Unless so much is given that one can directly calculate from each equation, what sometimes is the case.)

We are only at the very beginning. We will study all this a lot more.

\(^3\)This is not always so simple. In this class, luckily we can always do that directly.
Motivation: how to describe the following:

- position in the room:
  - 20 ft...

- driving away from the ocean (and the tsunami):
  - coast line
  - along
  - away!

- Driving at speed \( v_0 \) does not get us away from the coast at that speed!

Coordinates (components) in two dimensions:

- \( (x, y) \) - Cartesian
- \( (r, \theta) \) - polar

Position:
- Distance from "wall1"
- Distance from "wall2"

Velocity:
- \( (V_{away}, V_{along}) \)
- \( (V_x, V_y) \)
we are talking about vectors, and their components (or: coordinates). How do we write this properly? Say, for position:

\[ \mathbf{r} = (x, y). \]

This is a way to properly describe a vector.

Or, one can 'build' a vector, by adding two vectors along coordinate axes:

Simple vector addition:

\[ \mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2. \]

The point is to introduce "unit vectors" (along \( x \) and \( y \)):

With this:

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j}. \]

(according to this drawing, 

\( x \) coordinate of \( \mathbf{r} \) is 

around 7.5, and \( y \) is \( \approx 5 \).)
This holds for any vector (not only position),
for example,
\[
\vec{v} = (v_x, v_y) = v_x \hat{i} + v_y \hat{j}.
\]
These are simply different ways to write it.

Relation between components and magnitude ("length"):

\[
\vec{A} = (A_x, A_y) \quad \text{or} \quad \vec{A} = A_x \hat{i} + A_y \hat{j}
\]

\[
A_x = A \cos \Theta \\
A_y = A \sin \Theta
\]

"Magnitude" of a vector, \( |\vec{A}| \) is often written simply as: \( A \). It is strictly positive.

From the triangle above:

\[
|\vec{A}| = + \sqrt{A_x^2 + A_y^2}
\]

Also, since \( \tan \Theta = \frac{A_y}{A_x} \) \( \Rightarrow \) \( \Theta = \tan^{-1} \left( \frac{A_y}{A_x} \right) \).

\[
A_x = A \cdot \cos \Theta, \quad A_y = A \cdot \sin \Theta \quad \text("A" \equiv |\vec{A}|).
\]

This is (almost) all we'll need!
NOT actually done in class
(will be done next time):

how is this angle measured? Or, when
are components positive/negative?

1. Components are **always**:
   \[ A_x = +A \cos \theta, \quad A_y = +A \sin \theta, \quad \text{BUT} \]

2. The angle goes from "positive \( x \)"

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For example,
\[ \theta = 45^\circ \]

\[ A_x = A \cos \theta \]

This is positive!

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Negative, isn't it?
Cosine will give us this!

\[ A_x = A \cos \theta \text{ and } A_y = A \sin \theta \]

Try: use, for example, \( \theta = 330^\circ \).
Motivation: throw. what happens?

Start simpler: throw it horizontally.

Choose a coordinate system. (we need 2 coordinates)

The big thing: we can study the motion components (horizontal and vertical) SEPARATELY from each other!

This means that we can write down kinematics equations for the "x-part" of the motion, and separately for the y-component.
First: there are no forces acting horizontally, so the horizontal component of the velocity does not change; gravity is acting vertically, so the vertical component of the speed is subject to the acceleration of \( g = 9.8 \text{ m/s}^2 \) downward.

In other words, the acceleration vector* is given by:

\[
\begin{align*}
A_x &= 0, \\
A_y &= -g
\end{align*}
\]

* Since position and velocity are vectors in two dimensions (need two numbers to specify them), so is the acceleration.

We need Newton's laws to understand this fully. For now, let us work with it.
Note: In two dimensions:

→ position is a vector — need two coordinates

→ velocity is a vector, to specify it too: remember the driving-away-from-coast example? Understood differently: need to know both the 'actual' speed, AND the direction!

→ acceleration is the change of velocity ... so it must be a vector!

Then, we have the following conditions:

→ acceleration: \( a_x = 0, \quad a_y = -g \)

→ velocity: \( v_{0x} = v_0, \quad v_{0y} = 0 \)

we threw it horizontally:

\[
\begin{align*}
v_{0x} &= v_0 \\
v_{0y} &= 0
\end{align*}
\]
So:

using kinematics equations for x and y components of motion:

\[(x)\] \[x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \rightarrow \]
\[v_x = v_{0x} + a_xt \rightarrow \]

\[(y)\] \[y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 \rightarrow \]
\[v_y = v_{0y} + a_yt \rightarrow \]

\[\begin{align*}
X &= v_0 t \\
U_x &= v_0 \\
y &= -\frac{1}{2}gt^2 \\
U_y &= -gt.
\end{align*}\]

So we have the equations of motion, for this situation, in our coordinate system.

They describe: the position \((x, y)\) and the velocity \((v_x, v_y)\) of the object we threw out, at every point in time. We look at these in detail the next class, and solve for specific questions.