Solutions for HW #2.

These are solutions for homework problems for the second week of classes, 06/29–07/02.¹ All problems are from Knight’s book (2nd Edition), from Chapter 3: 44, and Chapter 4: 46, 47, 49.

Note the general kinematics equations. They are written for \( x \); the same set holds for \( y \) coordinate.

\[
\begin{align*}
x &= x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \\
v_x &= v_{0x} + a_xt \\
v_x^2 &= v_{0x}^2 + 2a_x\Delta x, \quad \text{where } \Delta x = (x - x_0)
\end{align*}
\]

We get equations of motion for the problem we study by applying these general equations to the problem, in our chosen coordinate system. I will use these freely, without specifically quoting them.

1 Ch3, problem 44.

This problem invokes intuitive understanding of forces: the three forces pulling on the knot need to add to zero. But here one should work purely with vectors: we need to find the third vector such that the sum of the three is zero. To add the vectors we add their components:

\[
\begin{align*}
\overrightarrow{V_1} + \overrightarrow{V_2} + \overrightarrow{V_3} &= \overrightarrow{U} \quad \Leftrightarrow \\
V_{1x} + V_{2x} + V_{3x} &= U_x \\
V_{1y} + V_{2y} + V_{3y} &= U_y
\end{align*}
\]

Need: \( \overrightarrow{U} = 0 \iff U_x = U_y = 0 \)  (1.1)

Recall that the \( x \) and \( y \) components of a vector \( \overrightarrow{V} \) are: \( V_x = V\cos \theta \) and \( V_y = V\sin \theta \), and the above equations, and the requirement of the problem (\( \sum_i \overrightarrow{V_i} = 0 \)), can be written out explicitly:

\[
\begin{align*}
V_1 \cos \theta_1 + V_2 \cos \theta_2 + V_3 \cos \theta_3 &= 0 \quad (V_1 = 3 \text{ units}, \ V_2 = 5 \text{ units}) \\
V_1 \sin \theta_1 + V_2 \sin \theta_2 + V_3 \sin \theta_3 &= 0 \quad (\text{angles given on the figure})
\end{align*}
\]

I use labels \( V_1 \equiv |\overrightarrow{V_1}| \), and \( V_2 \equiv |\overrightarrow{V_2}| \) etc, for magnitudes of vectors; the angles for \( V_1 \) and \( V_2 \) are given on the figure. Now we have a system of two equations, with two unknowns: \( V_3 \) and \( \theta_3 \), and we can solve that. (Or one could use \( V_{3x} \) and \( V_{3y} \) as unknowns. See Problem 4.) In principle, there may be an issue of how to actually use given angles to compute components: what are the signs of terms? Here we are given the full angles, from the positive \( x \)-axis, so we just use the definition: the components add, and the trig functions of the angles (\( \cos \theta \) and \( \sin \theta \)) will provide correct signs.

Here the ‘full’ angles are: \( \theta_1 = 0, \ \theta_2 = 120^\circ \), \( \theta_3 \) is unknown); the Eq.s (1.2)–(1.3) are:

\[
\begin{align*}
V_1 + \overrightarrow{V_2 \cos 120^\circ} + \overrightarrow{V_3 \cos \theta_3} &= 0 \quad (1.4) \\
0 + \overrightarrow{V_2 \sin 120^\circ} + \overrightarrow{V_3 \sin \theta_3} &= 0 \quad (1.5)
\end{align*}
\]

¹Posted on July 10, 2009. Minor modifications: July 17, 2009. (And a number in problem 2 corrected.)
To solve Eqs (1.4–(1.5)) for \(V_3\) and \(\theta\), use a little trick to avoid some algebra. Rewrite them as:

\[
\begin{align*}
V_3 \cos \theta_3 &= -V_1 - V_2 \cos 120^\circ \\
V_3 \sin \theta_3 &= -V_2 \sin 120^\circ
\end{align*}
\]

(1.6)

(1.7)

Dividing these equations now results in an equation where \(V_3\) cancels. Divide (1.6) by (1.7):

\[
\frac{V_3 \cos \theta_3}{V_3 \sin \theta_3} = \frac{-V_1 - V_2 \cos 120^\circ}{-V_2 \sin 120^\circ} \quad \Rightarrow \quad \cot \theta_3 = \frac{V_1 + V_2 \cos 120^\circ}{V_2 \sin 120^\circ} = 0.115
\]

(1.8)

Inverse cotangent will give us the acute angle; we must realize, from the physics of our problem, that our angle has to be in the 3rd quadrant, and so we must add \(\pi\): \(\theta_3 = 180^\circ + 83.4^\circ = 263^\circ\). (This angle satisfies \(\cot \theta_3 = 0.115\) too.) Now use either equation to compute \(V_3\); from Eq. (1.7):

\[
V_3 = \frac{-V_2 \sin 120^\circ}{\sin \theta_3} = 4.36 \text{ (units)}
\]

Note that in the setup we never had to think about signs, or where the angles are, or such.

Another way is to determine, from our physics, what the signs of the components are, and to use the simplest, acute, angles. (Often this is how the angles are given.) Here this would be:

Here the angles we use are: \(\alpha_1 = 0\), \(\alpha_2 = 60^\circ\),

but only given that we decide about signs:

\[
\begin{align*}
V_1 - V_2 \cos 60^\circ - V_3 \cos \alpha_3 &= 0 \\
0 + V_2 \sin 60^\circ - V_3 \sin \alpha_3 &= 0
\end{align*}
\]

Proceed to solve the system for \(V_3\) and \(\alpha_3\).

Often this is easier. But beware: definitions of components now do not prevent mistakes in signs.

Note that the term \((+V_2 \cos 120^\circ)\) we had when using the definition of adding components is of course equal to the appropriate term above, \((-V_2 \cos 60^\circ)\); and that \((V_2 \sin 120^\circ) = (+V_2 \sin 60^\circ)\).

2 Ch4, problem 46.

We will need to find the vertical coordinate of the ball when it has the same horizontal coordinate as the net does; then we can compare it with the height of the net. It may make equal sense to use either of these coordinate systems: tied to the ground, or to the level of the net, or to the level the ball is shot from. (Just make sure you show your choice.) I will use the one on the sketch:

The equations of motion are:

\[
\begin{align*}
x &= (v_0 \cos \theta) t \\
y &= (v_0 \sin \theta) t - \frac{1}{2} gt^2
\end{align*}
\]

(2.1)

(2.2)

Note that the velocity component equations are left out, since we don’t need them here.
Once the ball is above the net (or hitting it), we know its \( x \)-coordinate: it is the position of the net (call it \( x_f \)). We can use this to find the time when it is there \( (t_f) \), from Eq. (2.1); then use this time in Eq. (2.2) to get the ball’s \( y \)-coordinate \( (y_f) \):

\[
\text{Ball at net’s } x \text{-coordinate: } \quad x_f = v_0 \cos \theta t_f \quad \Rightarrow \quad t_f = \frac{x_f}{v_0 \cos \theta} \quad (= 0.351 \text{ s}) \quad \text{Then:}
\]

\[
y_f = v_0 \sin \theta \left( \frac{x_f}{v_0 \cos \theta} \right) - \frac{1}{2} g \left( \frac{x_f}{v_0 \cos \theta} \right)^2, \quad y_f = x_f \tan \theta - \frac{g x_f^2}{2 v_0^2 \cos^2 \theta} = 0.01 \text{ m}
\]

Since in my coordinates the net’s top is at \( y = -1 \text{ m} \), the ball clears it by 1.01 m.

### 3 Ch4, problem 47.

Equations of motion are different for parts (a) and (b). Or, one can solve part (b), and use \( \theta = 0 \) in its solution, thus getting to part (a). To review the horizontal shot, I will solve them separately.

**part (a)** Here we have a ‘horizontal shot:’ the initial velocity has no vertical component. This makes the equations of motion much simpler: \( v_0 \cos \theta = v_0 \cos 0 = v_0 \), and \( v_0 \sin \theta = v_0 \sin 0 = 0 \).

Our equations of motion:

\[
x = v_0 t \quad (3.1)
\]

\[
y = -\frac{1}{2} gt^2 \quad (3.2)
\]

(Also: \( v_x = v_0 \) and \( v_y = -gt \).)

Note Eq. (3.1): we know the \( x \)-coordinate of the point where the ball hits the ground; if we find the time for this, then we can solve for \( v_0 \). Time is found via Eq. (3.2): when the ball hits the ground, we know its \( y \)-coordinate, \( y_f = -H \) (in our chosen system), and with this we get the time:

\[
\text{At ground: } \quad y_f = -H = -\frac{1}{2} gt_f^2 \quad \Rightarrow \quad t_f^2 = \frac{2(-H)}{g} \quad \Rightarrow \quad t_f = \pm \sqrt{\frac{2H}{g}} \quad (3.3)
\]

We clearly need the plus sign. Now we can use Eq. (3.1), along with the known \( x_f \), to find \( v_0 \):

\[
x_f = v_0 t_f \quad \Rightarrow \quad v_0 = \frac{x_f}{t_f} = \frac{L}{\sqrt{\frac{2H}{g}}} \quad \Rightarrow \quad v_0 = L \sqrt{\frac{g}{2H}} = 27.7 \text{ m/s} \quad (3.4)
\]

This is an opportunity to make a note on rounding of results. If calculated with \( g \approx 10 \text{ m/s}^2 \), the above is: \( v_0 = 27.9508 \text{ m} \). In principle, rounding of final results relates to our estimate of their accuracy. Say that I have reasons to believe that this is reliable to the first decimal place; then I round to it: \( v_0 = 28.0 \text{ m/s} \) – and keep the zero. Having it written communicates that I consider that particular number as meaningful, and want to show that it is a zero, not any old number. The same holds if for any reason one needs a specific number of significant figures (when required, for example); the last one is shown, even if it is zero. I will generally keep 3 significant figures.
part (b) For this part of the problem \( v_{0y} \neq 0 \), and this is a full projectile problem.

Our equations of motion:

\[
\begin{align*}
  x &= v_0 \cos \theta \ t \\
  y &= v_0 \sin \theta \ t - \frac{1}{2} gt^2
\end{align*}
\]

And: \( v_x = v_0 \cos \theta \), \( v_y = v_0 \sin \theta - gt \).

Note that by allowing \( \theta \) to be either positive or negative we take into account both cases \( \theta = \pm 5^\circ \) by the above equations: the ball thrown either above or below the horizontal. Now when we apply these equations to the point on the ground, for which we know both coordinates, both will have the unknown \( v_0 \) (as well as \( t \)), so we cannot do exactly the same thing that we did in part (a). But we do get a system of two equations with two unknowns, and this is solvable. So for the point on the ground where the ball lands:

\[
\begin{align*}
  x_f &= v_0 \cos \theta \ t_f \\
  y_f &= v_0 \sin \theta \ t_f - \frac{1}{2} gt_f^2
\end{align*}
\]

Eliminate \( t_f \): solve for it from the first equation (easier), and substitute that expression in the second. We get one equation with the only unknown being \( v_0 \), and we can solve for it. Then, substituting the positive and negative values for \( \theta \) we’ll get the limiting values for \( v_0 \), and thus the range of speeds.

\[
\begin{align*}
  x_f = v_0 \cos \theta \ t_f \quad &\Rightarrow \quad t_f = \frac{x_f}{v_0 \cos \theta}, \quad \text{and} \\
  y_f &= v_0 \sin \theta \left( \frac{x_f}{v_0 \cos \theta} \right) - \frac{1}{2} g \left( \frac{x_f}{v_0 \cos \theta} \right)^2 = \tan \theta \ x_f - \frac{1}{2} \frac{g \ x_f^2}{v_0^2 \cos^2 \theta}
\end{align*}
\]

We can now solve this for \( v_0 \), and use \( x_f = L \) and \( y_f = -H \)

\[
-H = \tan \theta \ L - \frac{1}{2} \frac{g \ L^2}{v_0^2 \cos^2 \theta} \quad \Rightarrow \quad v_0 = + \sqrt{\frac{g \ L^2}{2 \ (H + L \ \tan \theta) \ \cos^2 \theta}} \quad (v_0 \equiv |\vec{v}_0| > 0)
\]

(\( v_0 \) is the magnitude: need the + sign.) We nicely got both solutions in one formula. They are:

\[
\begin{align*}
  v_0^{(1)} &= \sqrt{\frac{g \ L^2}{2 \ [H + L \tan(5^\circ)] \ \cos^2(5^\circ)}} = 22.3 \text{ m} \\
  v_0^{(2)} &= \sqrt{\frac{g \ L^2}{2 \ [H + L \tan(-5^\circ)] \ \cos^2(-5^\circ)}} = 41.3 \text{ m}
\end{align*}
\]

These make sense: if thrown at same speeds, the downward shot would reach the ground sooner, and so it wouldn’t make it as far away horizontally as the upward one; so, to fall at the same place, its speed has to be greater. Another check: the speed of the shot with \( \theta = 0 \) is within this range.

Finally: the result in Eq. (3.11) with \( \theta = 0 \) does give the solution of part (a), as it should.
4 Ch4, problem 49.

First we need to sort out all those given distances.

\[ H = (H_1 - h) + H_2 = (H_1 - h) + \frac{L_1}{2} \tan \theta \quad (= 5 \text{ m}) \]
\[ L = L_1 + \frac{L_1}{2} \quad (= 9 \text{ m}) \quad (4.1) \]

Now we have clear data to work with. Tie the coordinate system to where the ball is thrown from.

We need \( v_0 \) and \( \theta \). These kinds of problems can be among the most complicated in kinematics.

Note, we are given two kinds of data – related to the ball’s landing spot, and to its highest point:

1. Distance to where it falls; and we know the \( y \)-coordinate of that spot (\( = 0 \)).

2. Both coordinates of its highest point; and we know that \( v_y = 0 \) there.

These two points are not directly related, except that: we do know that the time to land is twice the time it takes it to get to its maximal height. This intuitive observation can be supported too: the motion is described by the parabola, which is symmetric around its highest point; so its length to the symmetric points on both sides is equal – and the parameter along the curve, measuring its length, is precisely the time. So the time to ground is twice the time to top. This is an involved argument! And it is a special case, which holds only for symmetric points. I won’t use it.
This still looks like a lot of information, doesn’t it. It is, and this problem can be solved in a few ways. I will use the information about the highest point; and then solve the obtained equations in two ways. Apply equations of motion, knowing \( y_t, x_t \), and that \( v_y^{(t)} = 0 \) (index “t” is for “top”)

\[
x_t = v_0 \cos \theta \ t_t \\
y_t = v_0 \sin \theta \ t_t - \frac{1}{2} gt_t^2 \\
0 = v_0 \sin \theta - gt_t
\]  

This is a system of three equations with three unknowns: solvable! We are after \( v_0 \) and \( \theta \), and we don’t care for \( t_t \); solve for it in Eq. (4.4), and substitute this into the other two. Then we’ll have a system of two equations with two unknowns. So, \( t_t = v_0 \sin \theta / g \), and substituting:

\[
x_t = v_0 \cos \theta \left( \frac{v_0 \sin \theta}{g} \right) \quad \Rightarrow \quad x_t = \frac{v_0^2}{g} \sin \theta \cos \theta \tag{4.5}
\]

\[
y_t = v_0 \sin \theta \left( \frac{v_0 \sin \theta}{g} \right) - \frac{1}{2} g \left( \frac{v_0 \sin \theta}{g} \right)^2 \quad \Rightarrow \quad y_t = \frac{1}{2} \frac{v_0^2}{g} \sin^2 \theta \tag{4.6}
\]

To solve this we want to eliminate \( v_0 \); dealing with the trigonometric functions would clearly be much messier. And, there is a shortcut too: divide these equations with each other, and \( v_0 \) cancels. (And as for the trig, we can see that we’ll end up with a tangent or cotangent; and that is good.) Dividing, say, the second equation by the first:

\[
\frac{y_t}{x_t} = \frac{1}{2} \frac{v_0^2}{g} \sin^2 \theta \cos \theta \quad \Rightarrow \quad \frac{1}{\sin \theta \cos \theta} \quad \text{so:} \quad \tan \theta = \frac{2 y_t}{x_t} = \frac{2 H}{L} = 1.11 \quad \Rightarrow \quad \theta = 48.0^\circ
\]

Well, that was nice: not much work to get the angle! Note that when dividing equations you must divide the whole sides as they stand, and this is not always so useful. Above we had compact expressions on each side (one term), and got clean cancellations. With the angle, we can now go back to either of Eqs. (4.5)–(4.6) to get the speed. Using Eq. (4.6): \( v_0 = \sqrt{2gy_t / \sin^2 \theta} = \sqrt{2gH / \sin \theta} = 13.4 \text{ m/s}. \)

If we need symbolic solutions, we would have to somehow make use of the result (\( \tan \theta = 2H/L \)) above, for expressing \( \sin \theta \), in order to get \( v_0 \). This conversion is done by squaring the definition of tangent, then using \( \sin^2 \theta + \cos^2 \theta = 1 \) to express \( \cos \theta \) via \( \sin \theta \), and then solving for \( \sin \theta \):

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \Rightarrow \quad \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta}{1 - \sin^2 \theta} \quad \Rightarrow \quad \sin^2 \theta = \frac{\tan^2 \theta}{1 + \tan^2 \theta} \tag{4.7}
\]

Now we can use our result (the expression \( \tan \theta = 2H/L \)), in equations involving the \( \sin \theta \), like Eq. (4.6), without knowing the numbers. So the final (symbolic) solution is

\[
v_0 = \sqrt{\frac{2gy_t}{\sin^2 \theta}} = \sqrt{\frac{2gy_t}{\tan^2 \theta / (1 + \tan^2 \theta)}} \quad \Rightarrow \quad v_0 = \sqrt{\frac{2gH}{1 + \tan^2 \theta}} \frac{1 + \tan^2 \theta}{\tan^2 \theta}
\]

and we can now make use of our symbolic result \( \tan \theta = 2H/L \) to get a formula for \( v_0 \)

\[
v_0 = \sqrt{\frac{2gH}{(2H/L)^2}} = \sqrt{2gH} \frac{L^2 + 4H^2}{L^2} = \sqrt{\frac{2gH}{L^2}} \frac{L^2 + 4H^2}{4H^2} = \sqrt{g} \frac{L^2 + (2H)^2}{2H} \tag{4.7}
\]

This form allows us to see the effect of the geometry of our problem more clearly, for example.
Now let’s see another, very handy, way to find solutions to the above system, Eq.s (4.2)-(4.4), bypassing trigonometry altogether. Write the initial velocity components as such:

\[
x_t = v_{0x} t_t \\
y_t = v_{0y} t_t - \frac{1}{2} g t_t^2 \\
0 = v_{0y} - g t_t
\]  

(4.8) \hspace{1cm} (4.9) \hspace{1cm} (4.10)

Voila: instead of \(v_0\) and \(\theta\), now we have variables \(v_{0x}\) and \(v_{0y}\), the components of the initial velocity – and no angles are involved. (Also, sometimes one may want precisely these.) Once we solve this system for \(v_{0x}\) and \(v_{0y}\) (after eliminating the time), we can get final answers as: \(v_0 = \sqrt{v_{0y}^2 + v_{0x}^2}\), and \(\tan \theta = v_{0y} / v_{0x}\). With \(t_t = v_{0y} / g\), from the third equation above, the first two become

\[
x_t = v_{0x} \left( \frac{v_{0y}}{g} \right) \\
y_t = v_{0y} \frac{v_{0y}}{g} - \frac{1}{2} g \left( \frac{v_{0y}}{g} \right)^2, \hspace{1cm} y_t = \frac{1}{2} \frac{v_{0y}^2}{g}
\]  

(4.11) \hspace{1cm} (4.12)

And it turns out that we don’t have to ‘solve the system’ – we can get \(v_{0y}\) directly from the second equation: \(v_{0y} = \sqrt{2g} y_t\). Having this we use it in the first equation to get \(v_{0x} = g x_t / \sqrt{2g} y_t\). Now we can use these components of the vector \(\vec{v}_0\), to find its magnitude and angle:

\[
v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} = \sqrt{2g y_t + \frac{g x_t^2}{2 y_t}} = \sqrt{2g H + \frac{gL^2}{2H}} = \sqrt{g \left( \frac{4H^2}{2H} + \frac{L^2}{2H} \right)} = \sqrt{g \frac{L^2 + 4H^2}{2H}}
\]

\[
\tan \theta = \frac{v_{0y}}{v_{0x}} = \left( \frac{\sqrt{2g} y_t}{g x_t} \right)^2 = \frac{2 y_t}{g x_t} = \frac{2 y_t}{x_t} = \frac{2H}{L}
\]

Note the utility of doing symbolic calculations: we can see that this solution matches the previous one without having any numbers.