Solutions\(^1\) for HW #4:  Ch6: 44, 46, 52; Ch7: 29, 41. (Knight, 2nd Ed).

We make use of: equations of kinematics, and Newton’s Laws. You also (routinely) need to handle components of a vector, in nearly every problem. Kinematics:

\[
\begin{align*}
    x &= x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \\
    v_x &= v_{0x} + a_xt \\
    v_x^2 &= v_{0x}^2 + 2a_x\Delta x, \quad \text{where } \Delta x = (x - x_0)
\end{align*}
\]

The above equations are written for \(x\); the same set holds for the \(y\) coordinate. Newton’s Laws:

\[
\sum_i F_i = m\ddot{a} \quad \Leftrightarrow \quad \sum_i F_{ix} = ma_x \quad \sum_i F_{iy} = ma_y
\]

For a vector \(\vec{V}\) components are: \(V_x = V\cos\theta\), and \(V_y = V\sin\theta\). (Here \(V \equiv |\vec{V}|\) is the magnitude.)

1 Ch6, problem 44.

This should be a fairly straightforward ‘inclined-plane problem,’ with a (typical) question on kinematics. We first need to find the acceleration; set up the Newton’s Laws:

\[
\begin{align*}
    (x) & \quad F_{\text{gr},x} + F_{\text{fr}} \sin(\theta) - F_{\text{fr}} = m a_x \\
    (y) & \quad -F_{\text{gr},y} \cos(\theta) + F_N = m a_y
\end{align*}
\]

Gravity’s \(x\) component is: \(F_{\text{gr},x} \cos(90^\circ - \theta)\), and we use trigonometry to express it via the angle that is given: \(\cos(90^\circ - \theta) = \sin(\theta)\). The same trick goes for its \(y\) component, use: \(\sin(90^\circ - \theta) = \cos(\theta)\).

Since it is important to get these things straight, let’s spell out the component business used above. The \(x\) component of the force of gravity is given by the cosine of the angle with the positive \(x\) axis. From the geometry of the problem I figured that this will be the (positive) cosine of the nearest acute angle, which is shown on the figure. Now, this is not the angle we are given \(\theta\); rather, it is equal to \((90^\circ - \theta)\). So, we have: \(F_{\text{gr},x} = F_{\text{gr}} \cos(90^\circ - \theta)\). This is where we use the trigonometry of angles in a right-triangle: \(\cos(90^\circ - \theta) = \sin(\theta)\). And we get: \(F_{\text{gr},x} = F_{\text{gr}} \cos(\theta)\). The same analysis goes for gravity’s \(y\)-component: \(F_{\text{gr},y} = -F_{\text{gr}} \sin(90^\circ - \theta) = -F_{\text{gr}} \cos(\theta)\). One does this mostly so that the components can be expressed conveniently via the angle that is ‘given’ only symbolically.

We need to solve the above system for \(a_x\). The reference frame is chosen so that the unknown \(\ddot{a}\) has only the \(x\) component: \(a_y = 0\). Making use of this, and of \(F_{\text{fr}} = \mu_k F_N\), \(F_{\text{gr}} = mg\), we have:

\[
\begin{align*}
    (x) & \quad +mg\sin(\theta) - \mu_k F_N = m a_x \\
    (y) & \quad -mg\cos(\theta) + F_N = 0
\end{align*}
\]

Eliminate $F_N$: solve for it from Eq. (1.4), and use this expression in Eq. (1.3)

$$F_N = mg \cos \theta \quad \Rightarrow \quad mg \sin \theta - \mu_k (mg \cos \theta) = ma_x$$  \hspace{1cm} (1.5)

Dividing through by $m$ we get the solution to our problem of dynamics:

$$a_x = g \sin \theta - \mu_k g \cos \theta, \quad \text{or,} \quad a_x = g (\sin \theta - \mu_k \cos \theta)$$  \hspace{1cm} (1.6)

Having this we can answer any kinematical questions (given enough information).

The problem asks for such initial velocity so that after a certain distance ($L$) the speed is zero. This is the reign of Eq. (k3), which becomes:

$$0 = v_{0x}^2 + 2a_x \Delta x, \quad \text{where} \quad \Delta x = (x - x_0) = L \quad \Rightarrow \quad v_{0x} = +\sqrt{-2a_x L}. \hspace{1cm} (1.7)$$

We chose the $+$ sign since the initial velocity should clearly be directed toward positive $x$, and so $v_{0x}$ is positive. From the statement of the problem we expect $a_x$ to be negative: you push a box, which slows down to a stop; thus the minus sign under the root is not (much) unnerving. Anticipating this sign, reverse the terms in $a_x$, pulling out the minus sign; our result is:

$$v_{0x} = \sqrt{2gL (\mu_k \cos \theta - \sin \theta)} \quad \text{with numbers:} \quad v_{0x} = 2.73 \text{ m/s} \quad (a_x = -0.743 \text{ m/s}^2) \hspace{1cm} (1.8)$$

Let us try out our formula. To do this we can, for example, pick some ‘special’ cases (or values) for the quantities involved in the formula, that yield distinct physical situations. We may go for the ones that we understand well, in which case we are checking our result; or, we may try the ones we are interested in learning about, in which case our result is put to even better use (than only for solving this particular problem).

Here the angle $\theta$ is one clear target for such games. With values such as $\pm 90^\circ$, $0$ etc, our result simplifies greatly. And it is clear what situations these values represent. When $\theta = 0$, we are talking about a flat surface; our result above becomes: $v_0 = \sqrt{2\mu_k g L}$. Remember stopping distances and such? This would be just such a setup, and this $v_0$ should be the answer to the question: ‘with what speed was I cruising if I stopped in distance $L$, given $\mu_k$’ (Check whether this would indeed be the correct answer.) For $\theta = 90^\circ$ ... huh, we get a negative number under the root! But think about it: $\theta = 90^\circ$ means we drop the thing straight down – it can’t slow down and stop! So this meaningless ‘result’ only reminds us that we cannot use the formula in Eq.(1.8) for such a case. But, try $\theta = -90^\circ$: now this is a shot straight up; then we get $v_0 = \sqrt{2gL}$. The fact that here $\mu_k$ dropped out makes sense: for the vertical motion there is no $F_N$, and so there is no friction; so $\mu_k$ doesn’t figure in the solution.

Let us look at another quantity that offers clear analysis: what is our result for a perfectly smooth surface, when $\mu_k = 0$? Once again we get a negative number under the root: without friction the box could not be slowing down – it would be accelerating (downward). So the condition of the problem $v_f = 0$ would not make sense, which is why ‘the result’ doesn’t.

Can you come up with some more discussion, checks, or other special cases that the above formula may cover?

I cannot emphasize enough just how important it is that you take a good, hard look at your results; and that you analyze them, push them and stretch them, try out the formulas you get.

Another comment. The above ‘negative acceleration’ isn’t luck: “if it were” positive, the system could not come to a stop; we would be talking about a different problem, and the Eq. (k3) would be different when applied to such a setup.
2 Ch6, problem 46.

Another very typical problem. Note that once the system starts coming down, the direction of friction is opposite from where it was while the block was coming up. So the forces are different, and we’ll have to set it up again. (We also need to check whether it gets stuck when it stops.)

When the block is moving up along the plane, the friction is directed down (along the plane)

\[ F_{fr} = \mu_k F_N \]
\[ F_{gr} = mg \]
\[ a_y = 0. \]

We need \( a_x \), so that we can answer the kinematical question. Using \( F_{fr} \), and \( F_{gr} \), and \( a_y = 0 \):
\[
\begin{align*}
(x) & \quad -mg \sin(\theta) - \mu_k F_N = ma_x \\
(y) & \quad -mg \cos(\theta) + F_N = 0
\end{align*}
\]

This system is solved for \( a_x \) by eliminating \( F_N \):
\[
F_N = mg \cos(\theta) \quad \Rightarrow \quad mg \sin(\theta) - \mu_k (mg \cos(\theta)) = ma_x
\]

and we get the solution for the motion of the system under consideration
\[
a_x = -g \sin(\theta) - \mu_k g \cos(\theta), \quad \text{or,} \quad a_x = -g (\sin(\theta) + \mu_k \cos(\theta)) \quad (2.6)
\]

The acceleration is directed down the plane (negative in our coordinates), while the system is moving up: it is slowing down. In doing so, both gravity and friction act in the same way, since their terms in the above result come with equal signs. This all makes sense. Now we can answer the question, using Eq. (k3), with \( v_f = 0 \), given \( v_0 \), and our result for \( a_x \):
\[
v_f^2 = 0 = v_0^2 + 2a_x \Delta x \quad \Rightarrow \quad \Delta x = -\frac{v_0^2}{2a_x} = \frac{v_0^2}{2g (\sin(\theta) + \mu_k \cos(\theta))} \quad (= 7.58 \text{ m}) \quad (2.7)
\]

The actual question is about the height, so a little trigonometry is due:
\[
\sin(\theta) = \frac{H}{\Delta x} \quad \Rightarrow \quad H = \Delta x \sin(\theta) = \frac{v_0^2}{2g} \frac{\sin(\theta)}{\sin(\theta) + \mu_k \cos(\theta)} = 3.79 \text{ m}. \quad (2.8)
\]

We must first check whether the block gets stuck when it stops: it doesn’t. See problem 3 for this. The motion is now down (the plane), so the friction is directed up the plane. Please note that the equations are set up exactly the same way as above, except that the force of friction acts the other way round. So now we have:
\[
\begin{align*}
(x) & \quad -mg \sin(\theta) + \mu_k F_N = ma_x \\
(y) & \quad -mg \cos(\theta) + F_N = 0
\end{align*}
\]
This is solved the same way, eliminating $F_N$, with the result: $a_x = -g (\sin \theta - \mu_k \cos \theta)$. Note the change in sign (compared to the first part of the motion). Now gravity and friction do not act in concert: while friction is still slowing down the system (positive direction in our frame), the gravity is now providing acceleration (in the negative direction). Based on the result of the previous part, we could have really simply written down this even without working it out.

We need the final speed. Again we can make use of Eq. (k3), and we have: $\Delta x = (-L-0) = -L$, where $L$ is the distance from the spot at which it stopped to the beginning of the plane (where it had started), calculated in the previous part, in Eq. (2.7); and $v_0 = 0$. Solving it for $v_f$:

\[
 v_f^2 = 2a_x \Delta x = 2( -g (\sin \theta - \mu_k \cos \theta))(-L), \quad \Rightarrow \quad v_f = -\sqrt{2gL(\sin \theta - \mu_k \cos \theta)} \quad (2.11)
\]

Now we can use the expression for the distance in Eq. (2.7), and with a little clean-up we get:

\[
 v_f = -\sqrt{\frac{v_0^2}{2g(\sin \theta + \mu_k \cos \theta)}} (\sin \theta - \mu_k \cos \theta) = -v_0 \sqrt{\frac{\sin \theta - \mu_k \cos \theta}{\sin \theta + \mu_k \cos \theta}} \quad (2.12)
\]

Of course, having calculated the distance ($L$) previously, we could just insert numbers in Eq. (2.11). But using the expression for $L$ we get a formula, that provides a lot more information. Firstly, it is clear that this speed is lesser than the initial $v_0$. (The denominator under the root is greater than the numerator, so the root will be of a number less than 1. Taking the root cannot ‘tip it over’ 1.)

The number for it is: $v_f = -6.97 \text{ m/s}$. That’s quite a bit less than what it started with.

Now look at this result some more. With cosine/sin, the first thing one can do is calculate for angles of $\pm 90^\circ$, and 0 (and 180° etc). With $\theta = 0$ we get a root of $-1$: this would be a flat surface, so the block would not slide down; the way we solved the problem would not be a valid physical representation of something sitting still, so of course we get non-sense. For $\theta = 90^\circ$, we get $v_f = -v_0$. It is a straight-down drop, and there is no friction; we get the same result with $\mu_k = 0$ in our formula. Thus with no friction the block returns with the same speed it had been thrown with. With friction, it comes back with less. Remember this result. It will make perfect sense in the context of energy considerations.

3. Ch6, problem 52.

Let’s talk about friction a little. Firstly: it is due to the interaction between surfaces, for which we use “normal force” ($F_N$). Note that this force may have nothing to do with gravity; while it may, in a specific situation, be caused by the weight of an object bearing down on a surface. In this example, the friction is due to the horizontal component of the force ($F_0$), which presses the object against the surface, causing the interaction (which results in friction). The force that thus presses on the surface is, in a manner of speaking, ‘half’ of the interaction; the other part being the force from the surface on the object, pushing the object back. This statement describes a model for an interaction via the 3rd Newton Law: there is a ‘pair of forces,’ acting with equal magnitude and opposite directions, each on one of the objects interacting. Please note that these forces from the pair act on different objects. Remember that the 3rd NL is, simply speaking, a way to model/represent interaction between different objects. This makes it clear that the forces from the pair will be equal (it’s one interaction!), and opposite (each object is acted on in the same way: both pulled or both pushed). Of course, the effect of this same force on the objects is in general different: under the same force different masses respond differently ($F = ma$), so they pick up different acceleration.

\footnote{This statement is really loose, and the description is an oversimplification.}
So the block is ‘pushed back’ by the wall, and we call this the “normal force,” \( F_N \) (or \( N \)). This is the force used to describe friction: \( F_t = \mu F_N \). Please note that there is the same force (as \( F_N \)) acting on the wall, which is the ‘other part’ of the interaction between the surfaces of the block and of the wall. Since we usually don’t study how the wall moves/behaves, this force mostly does not come into play.

Back to the problem: which way is static friction in this case? It may be that we are pushing just barely enough to hold the block in place; it is about to slip down. In this case, the friction would be upward, resisting the motion to happen (the trend of it). Or, maybe we are pushing almost hard enough to make the block accelerate upward: it is about to go up! Then the friction is opposing that motion (to be), so acting downward. And there is the ‘in between’ regime: when the force is between these two bordering values, of barely holding it, and of almost accelerating it up. So there is a range of forces (conditions) for which an object may not move. Put some numbers in, see how they work out; this is a good example for the issue of the direction of static friction.

There is a very practical way to decide: calculate how the thing moves with all forces just like in the problem, except for friction. Then the friction is the other way. Finally, note that the problem does not state that there is friction, but it does say ‘wood block’ and ‘wood wall.’ So we need to consider friction, and to look up coefficients for wood-on-wood (Table 6.1, on p.163).

Eq.s (NL) of Newton’s Laws in our coordinates:

\[
\begin{align*}
(x) & \quad -F_0 \cos(\theta) + F_N = m a_x \quad (= 0) \\
(y) & \quad +F_0 \sin(\theta) - F_{fr} = m a_y \\
\end{align*}
\]

The direction of friction need be found out by analysis: how would the system move without it? Note that we also have: \( a_x = 0 \); and \( F_t = \mu F_N \).

To find out which way the system would go without friction, set up Newton’s Laws, Eq.s (NL)

\[
\begin{align*}
(x) & \quad -F_0 \cos \theta + F_N = m a_x \quad (= 0) \\
(y) & \quad +F_0 \sin \theta - m g = m a_y \\
\end{align*}
\]

With no friction all we need is the vertical equation; from it:

\[
F_0 \sin \theta - m g = m a_y, \quad a_y = -g + \frac{F_0}{m} \sin \theta \sim -10 \text{m/s}^2 + \frac{12 \text{N}}{1 \text{kg}} \sin(30^\circ) = -4 \text{m/s}^2
\]

Or, we can find the \( y \) component of the total force: \( \sum_i F_{i,y} \sim -10 \text{N} + (12 \text{N}) \sin(30^\circ) = -4 \text{N} \), and since this is downward the system has to move that way. Either way: the box would be sliding down; and so the friction is upward. This is the reason behind the plus sign in Eq. (3.2).

One could now ‘normally’ proceed to solve for acceleration ... except that one question remains: it could be that \( F_0 \) is sufficient to keep the block in place, when friction is included. Then this will be static friction at work, and then \( a_y = 0 \).

If this is not the case, then we can indeed solve the above system for \( a_y \), with the kinetic friction acting (so, with \( \mu_k \)). But first we must check whether static friction can stop the slide. A complication is that the force of static friction develops only as much of a magnitude as necessary. We can deal with this in two ways.
Either: compute the acceleration with the $\mu_s$, thus effectively using $F_{fr}^{(\text{max})} = \mu_s F_N$; this way we are using the maximum possible value for static friction. If we get a positive number, this means that static friction certainly can hold it still; and so there is actually no motion. The ‘result’ we get (if positive) only tells us that the thing stays still – not that it actually started moving upwards! Friction cannot change the direction of motion.

Or: compute $y$ components of forces (which is what we have already done above), and compare this numerically to the maximum value of static friction. This way we get:

$$F_{fr}^{(\text{max})} = \mu_s F_N = \mu_s F_0 \cos \theta = 5.2 \text{ N} \quad \textgreater \quad \sum' F_{i,y} = 3.8 \text{ N (without friction)} \quad (3.6)$$

Static friction can build up even more strength than necessary. So the box does not move.

**But what if it were still sliding down ...**

Still, once we are at it, let’s see the solution of Eq.s (3.1-3.2), for downward motion with friction:

with $a_x = 0$, $F_{fr} = \mu_k F_N$: $F_N = F_0 \cos \theta \quad \Rightarrow \quad ma_y = \mu_k F_0 \cos \theta - mg + F_0 \sin \theta \quad (3.7)$

So we (would) have:

$$a_y = -g + \frac{F_0}{m} \sin \theta + \mu_k \frac{F_0}{m} \sin \theta \quad (3.8)$$

Note just how clearly we can see physical contributions from different forces, via the terms above. (Again: this is *not* our problem, we are merely using the setup, for a more complete example.)

**4 Ch7, problem 29.**

This is a basic example for how to deal with multiple objects, and with one kind of contact forces – those that ‘push.’ For an example of how to model ‘pulling’ interaction, see the next problem.

First, let us take note of the basic presumption: the coefficient of friction of the bottom block must be greater than the one for the top block, for them to move as one; $\mu_1 > \mu_2$. When this is the case the friction on the bottom block is greater, and so on its own it would accelerate less than the top block would. (You can check this by computing the acceleration.) Since the mass doesn’t affect the acceleration (in such a case), the bottom block would be slower, and so the top block will indeed push on it. The problem gives us: $\mu_1 = 0.2$ and $\mu_2 = 0.15$, so all is well.

Now let me review the forces indicated on the sketch on Fig.1. The top block bears down on the bottom one, pushing it; there is interaction between these objects, conducted at the surface of their contact. The bottom block finds that ‘something’ is pushing it down the plane, while the top block is finding that something is in its way, slowing it down. This can be modeled via the 3rd NL: there is a “pair of forces,” perpendicular to the surface of contact, equal and opposite in direction. One is acting on the bottom block, pushing it down the plane, and it will be in this block’s equations; the other acts on the top block, pushing it up the plane, and this one will enter top block’s equations. They are equal in magnitude. Note that these are ‘normal forces’ by their nature, but that they have nothing to do with friction that these blocks suffer with the plane’s surface.

Recall the basic premise of dealing with multiple objects: we take each separately, and set up its NL’s equations, as if there is nothing else in the Universe. (This also means that each gets its own

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3 By “basic” I mean: ‘important;’ or ‘in the core’ of things; or ‘essential’ ... etc. Not ‘simple’ or ‘easy.’
Problem 7.29: Modeling the interaction with the 3rd Newton Law.

(+) Whatever actually affects them, should be included via the forces that have been identified. Note that this brings forth the benefits of modeling their interaction via the 3rd Newton Law: the fact that the bottom block does complicate the life of the top one, is represented in the top block’s equations by the force \( F_{12} \), together with a prescription for its direction (up the plane). Eventually, we’ll also be able to use the fact that the ‘forces of the pair’ (the 3rd NL’s device to model the interaction) have equal magnitudes.

A word on the notation, used right below; and on what will be used later. Force of gravity on a particular object is written as \( F_{gr}^{(1)} \) (on object ‘1’); in the future, instead of this full but clumsy label, I will use the self-explanatory \( m_1g \). The interaction between blocks is a normal force; it is perpendicular to the surface of contact between them. Then \( F_{N2}^{2-1} \) is one such force, acting on object 1 by object 2; normally I will label it by \( F_{21} \). The same will go for the force that object 2 is ‘feeling,’ by object 1: \( F_{N1}^{1-2} \); in the future we’ll use \( F_{12} \). For the top block (object “1”):

\[
\begin{align*}
(x) & \quad - F_{gr}^{(1)} \sin \theta + F_{fr}^{(1)} - F_{N1}^{2-1} = m_1 a_{1x} \\
(y) & \quad - F_{gr}^{(1)} \cos \theta + F_{N1}^{(1)} = m_1 a_{1y} = 0
\end{align*}
\]

and for the bottom object (“2”)

\[
\begin{align*}
(x) & \quad - F_{gr}^{(2)} \cos \theta + F_{fr}^{(2)} - F_{N2}^{1-2} = m_2 a_{2x} \\
(y) & \quad F_{N2}^{(2)} - F_{gr}^{(2)} \sin \theta = m_2 a_{2y} = 0
\end{align*}
\]

Now rewrite this in a simpler notation, described above, and using \( F_{fr} = \mu F_N \). Also use the following physical input: since these objects affect each other, their accelerations are correlated, in some way. In this case, they literally move together, so their \( a_x \)’s must be equal in magnitude.\(^4\)

\(^4\)This may not always be the case; for example, in some setups with pulleys.
A careful look at the coordinate systems we chose for each object reveals that the sign of the accelerations has to be the same too; so \(a_{1x} = a_{2x}\); I will use \(a_s\) for it (“s” for system).

\[
\begin{align*}
(x) & \quad -m_1 g \sin \theta + \mu_1 F_{N1} - F_{21} = m_1 a_s \\ 
(y) & \quad -m_1 g \cos \theta + F_{N1} = 0
\end{align*}
\] (4.5)

and for the bottom object

\[
\begin{align*}
(x) & \quad -m_2 g \sin \theta + \mu_2 F_{N2} + F_{12} = m_2 a_s \\ 
(y) & \quad -m_2 g \cos \theta + F_{N2} = 0
\end{align*}
\] (4.6)

Note that this is a system of 4 equations with 4 unknowns, thus solvable. To solve it for acceleration, we need to: eliminate both \(F_N\)'s, from equations for each object; then we get two equations from which we can eliminate the interaction force between blocks, and so get acceleration. Using \(F_{N1} = m_1 g \cos \theta\) from Eq. (4.6) in Eq. (4.5), and \(F_{N2} = m_2 g \cos \theta\) in Eq. (4.7), we get:

\[
\begin{align*}
\text{(block 1)} & \quad -m_1 g \sin \theta + \mu_1(m_1 g \cos \theta) - F_{21} = m_1 a_s \\ 
\text{(block 2)} & \quad -m_2 g \sin \theta + \mu_2(m_2 g \cos \theta) + F_{12} = m_2 a_s
\end{align*}
\] (4.7)

The above two equations clearly model the interaction between these two blocks. Now we can also make use of the 3rd Newton Law, for calculations: \(F_{12} = F_{21}\). Please note that the opposite direction for the forces of this pair has been taken into account, when we figured out the signs of their components in respective coordinate systems. In other words: it is true that \(\vec{F}_{12} = -\vec{F}_{21}\), but we already used the fact that these are vectors directed opposite to each other; now we are using the equality of magnitudes, \(|\vec{F}_{12}| = |\vec{F}_{21}|\), which we call simply \(F_{12}\) and \(F_{21}\). So we can eliminate this from the above equations; the most straightforward way is to add equations. Below I use “LHS” to stand for “left-hand-side,” and “RHS” for “right-hand-side.”

\[
\begin{align*} 
\text{LHS of Eq. (4.9)} & = \left( -m_1 g \sin \theta + \mu_1(m_1 g \cos \theta) - F_{21} \right) + \left( -m_2 g \sin \theta + \mu_2(m_2 g \cos \theta) + F_{12} \right) \\
\text{RHS of Eq. (4.9)} & = (m_1 a_s) + (m_2 a_s) = (m_1 + m_2) a_s \\
\text{RHS of Eq. (4.10)} & = \text{LHS of Eq. (4.10)}
\end{align*}
\] (4.8)

Now \(-F_{21} + F_{12} = 0\), since \(F_{21} = F_{12}\); then we divide by \(m_1 + m_2\), extract \(g\), and get:

\[
a_s = g \frac{-m_1 \sin \theta + \mu_1 m_1 \cos \theta - m_2 \sin \theta + \mu_2 m_2 \cos \theta}{m_1 + m_2}
\] (4.9)

A little grouping of terms allows us to see physics a lot more clearly (and it is nicer):

\[
a_s = -g \frac{(m_1 + m_2) \sin \theta - (\mu_1 m_1 + \mu_2 m_2 \cos \theta)}{m_1 + m_2}
\] (4.10)

or, better yet:

\[
a_s = -g \frac{\mu_1 m_1 + \mu_2 m_2 \cos \theta}{m_1 + m_2} \quad (= -1.82 \text{ m/s}^2)
\] (4.11)
Acceleration is due to: gravity’s component along the plane, down (our $x$-axis is up the plane); and opposed by the friction(s), which come from the normal forces, that are the responses to gravity’s components perpendicular to the plane (thus the cosine). The way friction along surfaces with different coefficients is mixed up is what the fraction tells us. Now this yields to a lot of analysis easily. What if the masses are equal? We get an effective coefficient of friction, being an average: $(\mu_1 + \mu_2)/2$. What if friction coefficients are equal? We get the behavior of one object. (Of course!) What is it when $\theta = 0$; and $\theta = 90^\circ$? Flat; vertical. We then know how it should behave.

I very strongly suggest that you always analyze your results, in particular their special cases.

Now we can answer the specific question. With the known initial speed ($= 0$), and the final position ($x_f = -L$), the Eq. (k1) allows to solve for the time directly:

$$x_f = -L = \frac{1}{2} a_x t_f^2 \implies t_f = \sqrt{\frac{-2L}{a_S}} = 1.48 \text{ s}$$

This certainly doesn’t look unreasonable; the boxes are heavy, and there is not much friction. On the other hand, the inclination angle is fairly small, and 2 m isn’t exactly little; this is a bit quick. A true confidence in a result can only come from a thorough analysis of its symbolic expression.

5 Ch7, problem 41.

This is a very useful problem, summarizing a lot about how to work with multiple objects.

Let us review the basics first. We approach this problem by setting up NL’s separately for each object, and for each in its own coordinate system, that we are free to choose. (The important thing is that we must set up a coordinate system for each; one cannot just say ‘ah, I’ll use the same.’) The fact that they affect each other is included by identifying the force of tension along the cord connecting them. The tension in the cup’s system acts upward (along the positive direction of its $y$ axis); the tension in the book’s setup acts down the plane (in the negative direction along the $x$ axis in its frame). These tensions are equal in magnitude. This way we took into account the opposite sign in the 3rd Newton Law, as well as the equality of magnitudes: $F_{T1} = -F_{T2}$. On the sketch above these (equal) magnitudes are called simply $F_T$. 

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The next thing to resolve is the relationship between their accelerations, since they must be correlated. If the cup moves upward (along its $+y$), the book would go up the plane (along its $+x$). If the cup were to go down (along its $-y$), the book would have to be going down (along its $-x$) too. As the cord doesn’t stretch, they always cover equal distances in equal times, in the same direction along the appropriate axes in their respective coordinate systems ... so we have: $a_{y\text{ (cup)}} = a_{x\text{ (book)}}$. Since I labeled the cup as object “1” and the book as object “2,” this is: $a_{1y} = a_{2x}$. Let me call this $a_S$. Now we can write Eq.s (NL) of Newton’s Laws for each object in their coordinate systems.

For the cup:

\[
\begin{align*}
(x) & \quad (\text{no forces on the cup along this direction}) \\
(y) & \quad - m_1 g + F_T = m_1 a_{1y}
\end{align*}
\tag{5.1}
\]

and for the book:

\[
\begin{align*}
(x) & \quad - m_2 g \sin \theta - \mu_2 F_{N2} - F_T = m_2 a_{2x} \\
(y) & \quad - m_2 g \cos \theta + F_{N2} = 0
\end{align*}
\tag{5.2}
\tag{5.3}
\]

Eliminating $F_{N2}$ from the book’s equations, and using $a_{1y} = a_{2x} \equiv a_S$, we get the system:

\[
\begin{align*}
(x) & \quad - m_1 g + F_T = m_1 a_s \\
(x) & \quad - m_2 g \sin \theta - \mu_2 m_2 g \cos \theta - F_T = m_2 a_s
\end{align*}
\tag{5.4}
\tag{5.5}
\]

Note the clarity of physics in these equations. The cup is pulled down by gravity, while pulled up by the tension. The book is pulled down the plane by everything: gravity’s component, friction, tension; it is slowing down after being shoved up the plane. The tension pulling up on the cup, opposing the gravitational pull down, is due to the fact that the cup ‘feels’ all that is going on with the book; likewise, the book is feeling the pull of the cup, and the sign of the tension in its equations shows this. Note that we never had to choose the sign for the acceleration; this will come out of our equations. What we most certainly must do carefully is to figure out the exact relation between the accelerations, for the equations to be consistent. (Here it was: $a_{1y} = a_{2x}$. ) To solve this for acceleration we need to eliminate the tension; add the above equations:

\[
( - m_1 g + F_T ) + ( - m_2 g \sin \theta - \mu_2 m_2 g \cos \theta - F_T ) = ( m_1 a_s ) + ( m_2 a_s )
\tag{5.6}
\]

The tension goes away. With a little grouping of terms, we get

\[
a_s = - g \frac{m_1 + m_2 \sin \theta + \mu_2 m_2 \cos \theta}{m_1 + m_2} = -6.73 \text{ m/s}^2
\tag{5.7}
\]

(Not that far from the value of $g$; this is slowing down rapidly.) I suggest you analyze this expression, trying out various special cases; for example, related to the angle; friction; masses, and the relation between masses.

Less importantly, now we can answer the specific questions.
(a) The first one goes directly via Eq. (k3). We have \( v_0 \), and \( v_f = 0 \); then \( 0 = v_0^2 + 2a_x \Delta x \) gives:

\[
\Delta x = \frac{-v_0^2}{2a_x} = \frac{-v_0^2}{2 \left( -g \frac{m_1 + m_2 \sin \theta + \mu_2 m_2 \cos \theta}{m_1 + m_2} \right)}, \quad \Delta x = \frac{v_0^2}{2g} \frac{m_1 + m_2}{m_1 + m_2 \sin \theta + \mu_2 m_2 \cos \theta} \tag{5.8}
\]

Try out this result, to make sure it makes sense. For example: with \( m_1 = 0 \) and \( \theta = 0 \) it should give the stopping distance from \( v_0 \), on the flat surface with \( \mu_2 \). Etc. (Numerically, \( \Delta x = 0.669 \text{ m} \).)

(b) Is the maximal static friction sufficient to hold against all other forces? Once the book stopped, it can only slide back down, and so the friction is up the plane. With this change, we can use our NL’s equations for the previous work:

\[
\sum_{\text{without friction}} F_{i_x} = -m_1 g \sin \theta - F_T, \quad \text{and from the cup’s equation: } F_T = m_2 g \quad \text{(since now } a = 0)\]

So we need to compare the magnitudes of \( F_{x}^{(\text{net})} \) and \( F_{\text{fr,s}}^{(\text{max})} \), numerically:

\[
\left| F_{x}^{(\text{net})} \right| = \left| -{(m_1 g \sin \theta + m_2 g)} \right| = 11.48 \text{ N} \quad > \quad F_{\text{fr,s}}^{(\text{max})} = \mu_s m_2 g \cos \theta = 4.60 \text{ N}
\]

Oh yes, the book will slide down. (Should we blame the cup? Or what? The angle?)