MTH 306 Spring Term 2007
Lesson 2

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- Learn the algebraic and geometric descriptions of vectors
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- Learn the basic rules of vector algebra and their geometric interpretations
Vectors, Lines, and Planes

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- Be able to find the component and projection of one vector along another vector
- Learn the vector and scalar equations for lines and planes in 2-space and 3-space
Vectors in Euclidean 2-space and 3-space

- Vectors have both algebraic and geometric descriptions.
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- The algebraic description enables you to calculate easily.

By contrast, a number, also called a scalar, has only magnitude. It has no direction.

Force and velocity are vectors. Mass, speed, and temperature are scalars.
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Force and velocity are vectors.

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The Component Presentation of Vectors

\[
\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{in 2-dimensions},
\]
\[
\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad \text{in 3-dimensions}.
\]

The **magnitude (length)** of a vector \( \mathbf{a} \) is

\[
|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \quad \text{in 2-dimensions},
\]
\[
|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \text{in 3-dimensions}.
\]

Vectors are added, subtract, and multiplied by scalars componentwise. In 3-dimensions, if

\[
\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad \text{and} \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle,
\]

and \( c \) is a scalar, then

\[
\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle
\]
\[
ca = \langle ca_1, ca_2, ca_3 \rangle.
\]
## Vector Arithmetic

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1.</td>
<td>( a + b = b + a )</td>
</tr>
<tr>
<td>A2.</td>
<td>( a + (b + c) = (a + b) + c )</td>
</tr>
<tr>
<td>A3.</td>
<td>( a + 0 = a )</td>
</tr>
<tr>
<td>A4.</td>
<td>( a + (-a) = 0 )</td>
</tr>
<tr>
<td>A5.</td>
<td>( c(a + b) = ca + cb )</td>
</tr>
<tr>
<td>A6.</td>
<td>( (c + d)a = ca + da )</td>
</tr>
<tr>
<td>A7.</td>
<td>( (cd)a = c(da) )</td>
</tr>
<tr>
<td>A8.</td>
<td>( 1a = a )</td>
</tr>
</tbody>
</table>

Why are these rules true?
Vector addition and subtraction can be visualized geometrically through a triangle law and a parallelogram law.
Geometric Presentation of Vectors

- Vector addition and subtraction can be visualized geometrically through a **triangle law** and a **parallelogram law**.
- Scalar multiplication scales the length of a vector and changes its direction when the scalar is **negative**.
A unit vector is a vector that has length one.
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If \( \mathbf{a} \neq 0 \), then \( \mathbf{u} = \mathbf{a} / |\mathbf{a}| \) is a unit vector with the same direction as \( \mathbf{a} \).
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In 3-dimensions, the unit vectors

\[
\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle
\]

provide a convenient alternative to the angle-bracket notation for components

\[
\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}
\]
A unit vector is a vector that has length one.

If \( a \neq 0 \), then \( u = a / |a| \) is a unit vector with the same direction as \( a \).

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a = \langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}
\]

The vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are called the standard basis vectors for 3-space.
An $n$-dimensional vector $\mathbf{a}$ has $n$ components

$$\mathbf{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle$$
An *n*-dimensional vector $\mathbf{a}$ has $n$ components

$$\mathbf{a} = \langle a_1, a_2, a_3, ..., a_n \rangle$$

Such vectors are added, subtracted, and multiplied by scalars componentwise.
N-dimensional Vectors

- An $n$-dimensional vector $\mathbf{a}$ has $n$ components

$$\mathbf{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle$$

- Such vectors are added, subtracted, and multiplied by scalars componentwise.

- Consequently, the algebraic properties A1-A8 hold for vectors in $n$-dimensions.
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The components of a vector can be either real or complex numbers.
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Consequently, the algebraic properties A1-A8 hold for vectors in $n$-dimensions.

The components of a vector can be either real or complex numbers.

The magnitude or length of $\mathbf{a}$ is

$$|\mathbf{a}| = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$
Vocabulary

- \( \mathbb{R}^n \) is the space of all \( n \)-vectors with real components and in which the scalars are real numbers
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- $\mathbb{R}^n$ is the space of all $n$-vectors with real components and in which the scalars are real numbers
- $\mathbb{C}^n$ is the space of all $n$-vectors with complex components and in which the scalars are complex numbers
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- $\mathbb{C}^n$ is the space of all $n$-vectors with complex components and in which the scalars are complex numbers
- The **standard basis in $n$-space** consists of the unit vectors
  - $\mathbf{e}_1 = \langle 1, 0, 0, \ldots, 0, 0 \rangle$
  - $\mathbf{e}_2 = \langle 0, 1, 0, \ldots, 0, 0 \rangle$
  - $\mathbf{e}_3 = \langle 0, 0, 1, \ldots, 0, 0 \rangle$
  - $\vdots$
  - $\mathbf{e}_n = \langle 0, 0, 0, \ldots, 0, 1 \rangle$
Vocabulary

- \( \mathbb{R}^n \) is the space of all \( n \)-vectors with real components and in which the scalars are real numbers.
- \( \mathbb{C}^n \) is the space of all \( n \)-vectors with complex components and in which the scalars are complex numbers.
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  \mathbf{e}_1 = \langle 1, 0, 0, \ldots, 0, 0 \rangle , \\
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  \mathbf{e}_3 = \langle 0, 0, 1, \ldots, 0, 0 \rangle , \\
  \vdots \\
  \mathbf{e}_n = \langle 0, 0, 0, \ldots, 0, 1 \rangle .
  \]

- Each vector \( \mathbf{a} \) in \( n \)-space can be represented as
  
  \[
  \mathbf{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 + \cdots + a_n \mathbf{e}_n .
  \]
For vectors in 2-space and 3-space the **dot product** of \( \mathbf{a} \) and \( \mathbf{b} \) is

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \cos \theta
\]

where \( \theta \) is the smaller angle formed by \( \mathbf{a} \) and \( \mathbf{b} \), so that \( 0 \leq \theta \leq \pi \).

The **component of \( \mathbf{b} \) along \( \mathbf{a} \)** is

\[
\text{comp}_\mathbf{a} \mathbf{b} = |\mathbf{b}| \cos \theta = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}
\]

**What are the geometric interpretations?**
For vectors in 2-space and 3-space the **dot product** of \( \mathbf{a} \) and \( \mathbf{b} \) is

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- **The component of \( \mathbf{b} \) along \( \mathbf{a} \)** is

\[
\text{comp}_a \mathbf{b} = |\mathbf{b}| \cos \theta = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}
\]

- **The projection of \( \mathbf{b} \) along \( \mathbf{a} \)** is

\[
\text{proj}_a \mathbf{b} = (\text{comp}_a \mathbf{b}) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a}.
\]

What are the geometric interpretations?
Finding the Angle Between Vectors

It is easy to find the angle $\theta$ between two nonzero vectors using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \quad \text{with} \quad 0 \leq \theta \leq \pi$$

Important:

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$$
Dot Product – Algebraic View

In terms of components

\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \]
\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

in 2-dimensions,
in 3-dimensions.

Notice that

\[ \mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 \]

and likewise in 2 dimensions. The dot product is called a “product” because it satisfies:

| DP1. \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \) | DP2. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \) |
| DP3. \( \mathbf{0} \cdot \mathbf{a} = 0 \) | DP4. \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \) |
| DP5. \( d (\mathbf{a} \cdot \mathbf{b}) = (d\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (d\mathbf{b}) \) |
Dot Product in Real N-space

The **dot product** (also called **scalar product** or **inner product**) of

\[ \mathbf{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle \] and \[ \mathbf{b} = \langle b_1, b_2, b_3, \ldots, b_n \rangle \]

in \( \mathbb{R}^n \) is

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- Properties DP1-DP5 hold
Dot Product in Real N-space

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- Properties DP1-DP5 hold
- **(Schwarz Inequality)** For any two \( n \)-vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( n \)-space,

\[ |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| \cdot |\mathbf{b}| \]
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- The Schwarz inequality enables us to extend the notion of angle, component, and projection to \( n \)-space - read on!
There is a unique angle \( \theta \) with \( 0 \leq \theta \leq \pi \) such that

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}
\]

By definition \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \).
There is a unique angle $\theta$ with $0 \leq \theta \leq \pi$ such that
\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}
\]
By definition $\theta$ is the \textbf{angle between a and b}.

Thus
\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta
\]
just as for vectors in 2- and 3-space.
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Thus

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

just as for vectors in 2- and 3-space.

We say two $n$-vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (perpendicular) if their dot product is zero:

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By definition $\theta$ is the **angle between $\mathbf{a}$ and $\mathbf{b}$**.

Thus

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The component and projection of one vector along another are defined just as in 2- or 3-space

$$\text{comp}_\mathbf{a} \mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{a}||},$$

$$\text{proj}_\mathbf{a} \mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{a}||^2} \mathbf{a}.$$
Example

Use dot product calculations to verify the identity

\[ |a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2. \]

Then give a geometric interpretation of this result for vectors in \( \mathbb{R}^2 \).
Row and Column Vectors

Sometimes it is most useful to think of a vector as a “row” vector and sometimes as a “column” vector.

- **Row vectors** are expressed by

  \[ \mathbf{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle \quad \text{or} \quad \mathbf{a} = [a_1 \ a_2 \ a_3 \ \ldots \ a_n] \]

  using angle or square brackets.
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  using angle or square brackets.

- **Column vectors** are expressed by

  \[ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}. \]
A Line $L$ is determined by a point $P_0$ on it and a vector $\mathbf{v}$ parallel to it.
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$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad \text{for } -\infty < t < \infty.$$
Lines in Space

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- The triangle law of vector addition gives a vector equation for $L$:
  \[ \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad \text{for } -\infty < t < \infty. \]
- If $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, and $\mathbf{v} = \langle a, b, c \rangle$, equating corresponding components in the vector equation gives (scalar) parametric equations for $L$:
  \[
  \begin{cases}
    x = x_0 + at \\
    y = y_0 + bt \\
    z = z_0 + ct
  \end{cases}
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The coefficients of $t$ give the components of a vector parallel to the line $L$. 

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If $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, and $\mathbf{v} = \langle a, b, c \rangle$, equating corresponding components in the vector equation gives (**scalar**) **parametric equations** for $L$:

$$\begin{cases} 
x = x_0 + at \\
y = y_0 + bt \\
z = z_0 + ct 
\end{cases} \quad \text{for } -\infty < t < \infty.$$ 

The coefficients of $t$ give the components of a vector parallel to the line $L$. The vector equation is true in $n$-space. What about the scalar equation?
Example

Find scalar parametric equations for the line determined by the two points $(-3, -1, 2)$ and $(4, 3, -2)$. 
A plane $\Pi$ in space is determined by a point $P_0$ on it and a vector $\mathbf{N}$ perpendicular (normal) to it.
A plane \( \Pi \) in space is determined by a point \( P_0 \) on it and a vector \( \mathbf{N} \) perpendicular (normal) to it.

An equation for \( \Pi \) is

\[
\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.
\]
A plane $\Pi$ in space is determined by a point $P_0$ on it and a vector $N$ perpendicular (normal) to it.

An equation for $\Pi$ is

$$N \cdot (r - r_0) = 0.$$ 

If $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, and $N = \langle a, b, c \rangle$, evaluating the dot gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

as an equation for the plane $\Pi$. 

This equation may also be put in the form $ax + by + cz = d$ by combining the constant terms.
Planes in Space

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- If $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, and $\mathbf{N} = \langle a, b, c \rangle$, evaluating the dot gives
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- This equation may also be put in the form
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  by combining the constant terms.
- Reverse the steps to see that either of these equations has graph a plane with normal $\mathbf{N} = \langle a, b, c \rangle$. 

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  by combining the constant terms.
- Reverse the steps to see that either of these equations has graph a plane with normal $\mathbf{N} = \langle a, b, c \rangle$.
- In the case of the first equation, the plane passes through the point $(x_0, y_0, z_0)$.
Example

A plane in space contains the point \((5, 4, 1)\) and has normal direction parallel to the line through the points \((0, -2, 0)\) and \((11, 7, -5)\). Find an equation for the plane.