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- Be able to define linear independence and linear dependence of a set of vectors
- Interpret independence and dependence in geometric terms for vectors in 2- and 3-space
- Use the determinant test or solve an appropriate linear system of equations to determine linear dependence or independence of a set of vectors
- Be able to find a nontrivial linear combination among linearly dependent vectors
Our main goal is to determine when a set of vectors does or does not contain redundant information.
Redundant and Non-redundant Data Sets

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- In the former case, the vectors are called linearly dependent.
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- In the former case, the vectors are called linearly dependent.
- In the latter case, the vectors are linearly independent.
A finite sum of the form

\[ c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k \]

is called a **linear combination** of the vectors \( v_1, v_2, v_3, \ldots, v_k \).
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A linear combination is called **nontrivial** if at least one of the scalars is not zero.
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\[ c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k \]

is called a **linear combination** of the vectors \( v_1, v_2, v_3, \ldots, v_k \).

A linear combination is called **nontrivial** if at least one of the scalars is not zero.

The set of all linear combinations of \( v_1, v_2, v_3, \ldots, v_k \) is called the **span** of the vectors \( v_1, v_2, v_3, \ldots, v_k \).
Example

Describe the span of the following sets of vectors in $\mathbb{R}^3$.

(a) \[
\begin{bmatrix}
1 \\
4 \\
7
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
2 \\
5 \\
8
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 \\
4 \\
7
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
5 \\
8
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
-5 \\
-11 \\
-17
\end{bmatrix}
\]

What about redundancy in the two cases?
Linear Dependence and Independence for Three Vectors in $\mathbb{R}^3$

- $a, b, c$ are LD $\iff$ $a, b, c$ all lie in a plane in $\mathbb{R}^3$.

Why?
Linear Dependence and Independence for Three Vectors in $\mathbb{R}^3$

- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are LD $\iff$ $\mathbf{a}, \mathbf{b}, \mathbf{c}$ all lie in a plane in $\mathbb{R}^3$.
- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are LI $\iff$ the span of $\mathbf{a}, \mathbf{b}, \mathbf{and} \mathbf{c}$ is $\mathbb{R}^3$.

Why?
A set of vectors $v_1, v_2, v_3, \ldots, v_k$ is **linearly dependent (LD)** if there are scalars $c_1, c_2, \ldots, c_k$ *not all zero* such that
\[
c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k = 0.
\]
A set of vectors $v_1, v_2, v_3, \ldots, v_k$ is **linearly dependent (LD)** if there are scalars $c_1, c_2, \ldots, c_k$ *NOT ALL ZERO* such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k = 0.$$ 

If the vectors are in $\mathbb{R}^n$ we may say the vectors are **linearly dependent over $\mathbb{R}$**, emphasizing the scalars are real numbers.
A set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k \) is **linearly dependent** (LD) if there are scalars \( c_1, c_2, \ldots, c_k \ NOT ALL ZERO \) such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \cdots + c_k \mathbf{v}_k = \mathbf{0}.
\]

If the vectors are in \( \mathbb{R}^n \) we may say the vectors are **linearly dependent over** \( \mathbb{R} \), emphasizing the scalars are real numbers.

If the vectors are in \( \mathbb{C}^n \) we may say the vectors are **linearly dependent over** \( \mathbb{C} \), emphasizing the scalars are complex numbers.
A set of vectors $v_1, v_2, v_3, ..., v_k$ is **linearly dependent (LD)** if there are scalars $c_1, c_2, ..., c_k$ *NOT ALL ZERO* such that

$$c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_kv_k = 0.$$

- If the vectors are in $\mathbb{R}^n$ we may say the vectors are **linearly dependent over** $\mathbb{R}$, emphasizing the scalars are real numbers.
- If the vectors are in $\mathbb{C}^n$ we may say the vectors are **linearly dependent over** $\mathbb{C}$, emphasizing the scalars are complex numbers.
- Corresponding language is used for linear independence.
A set of vectors $v_1, v_2, v_3, \ldots, v_k$ is **linearly independent (LI)** if it is not linearly dependent. Consequently:

A set of vectors $v_1, v_2, v_3, \ldots, v_k$ is **linearly independent** if and only if the equation

$$c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_kv_k = 0 \Rightarrow c_1 = 0, \ c_2 = 0, \ c_3 = 0, \ldots, \ c_k = 0.$$
Any linear combination of vectors in real or complex $n$-space can be expressed as $Ac$ where
Any linear combination of vectors in real or complex $n$-space can be expressed as $Ac$ where $A$ is the matrix whose columns are the vectors in the linear combination.
Linear Combinations – revisited

- Any linear combination of vectors in real or complex $n$-space can be expressed as $A\mathbf{c}$ where
- $A$ is the matrix whose columns are the vectors in the linear combination
- and $\mathbf{c}$ is the column vector whose components are the corresponding scalar multiples of those vectors.
Any linear combination of vectors in real or complex $n$-space can be expressed as $Ac$ where

- $A$ is the matrix whose columns are the vectors in the linear combination
- and $c$ is the column vector whose components are the corresponding scalar multiples of those vectors.

In symbols

$$c_1v + c_2w + \cdots = Ac$$

where $A = [v, w, \ldots]$ and $c = [c_1, c_2, \ldots]$ and only a finite number of vectors and scalars are involved.
The equivalence

\[ c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \iff Ac = 0 \]

where \( A = [v_1, v_2, \ldots, v_k] \) and \( c = [c_1, c_2, \ldots, c_k] \) leads immediately to tests for LD and LI in terms of solving systems of linear equations:
The equivalence

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \mathbf{0} \iff A \mathbf{c} = \mathbf{0} \]

where \( A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k] \) and \( \mathbf{c} = [c_1, c_2, \ldots, c_k] \) leads immediately to tests for LD and LI in terms of solving systems of linear equations:

- \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are LD if and only if the homogeneous system \( A \mathbf{c} = \mathbf{0} \) has nontrivial solutions for \( \mathbf{c} \).

- \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are LI if and only if the homogeneous system \( A \mathbf{c} = \mathbf{0} \) has only the trivial solution \( \mathbf{c} = \mathbf{0} \).

Consequently, a set of \( n \), \( n \)-vectors is LI if and only if the determinant of the matrix whose columns are the given vectors is not zero.
The equivalence

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \mathbf{0} \iff A\mathbf{c} = \mathbf{0} \]

where \( A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k] \) and \( \mathbf{c} = [c_1, c_2, \ldots, c_k] \) leads immediately to tests for LD and LI in terms of solving systems of linear equations:

- \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are LD if and only if the homogeneous system \( A\mathbf{c} = \mathbf{0} \) has nontrivial solutions for \( \mathbf{c} \).
- \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are LI if and only if the homogeneous system \( A\mathbf{c} = \mathbf{0} \) has only the trivial solution \( \mathbf{c} = \mathbf{0} \).

Consequently, a set of \( n \), \( n \)-vectors is LI if and only if the determinant of the matrix whose columns are the given vectors is not zero.
The equivalence

\[ c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \iff A c = 0 \]

where \( A = [v_1, v_2, \ldots, v_k] \) and \( c = [c_1, c_2, \ldots, c_k] \) leads immediately to tests for LD and LI in terms of solving systems of linear equations:

- \( v_1, v_2, \ldots, v_k \) are LD if and only if the homogeneous system \( A c = 0 \) has nontrivial solutions for \( c \).
- \( v_1, v_2, \ldots, v_k \) are LI if and only if the homogeneous system \( A c = 0 \) has only the trivial solution \( c = 0 \).
- Consequently, a set of \( n, n \)-vectors is LI if and only if the determinant of the matrix whose columns are the given vectors is not zero.
Test the given set of vectors for LI or LD. If LD find a nontrivial linear combination of the vectors that has sum zero.

\[
\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}
\]
Example

Test the given set of vectors for LI or LD. If LD find a nontrivial linear combination of the vectors that has sum zero.

\[
\begin{bmatrix}
1 \\
1 \\
0 \\
-1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
1 \\
2
\end{bmatrix},
\begin{bmatrix}
0 \\
2 \\
-1 \\
1
\end{bmatrix}
\]