2 The Wave Equation

2.1 Wave Motion in One Spatial Dimension

Consider a homogeneous string (such as a guitar or violin string) at rest and tautly stretched between two posts:

the string has congruent cross sections each with area $A_0$
the string at rest has length $L$
the string at rest has constant density $\tilde{\rho}_0$ (mass/volume)

We set the string in motion, perhaps by plucking it, and wish to describe the subsequent motion of the string. We make the following basic assumption:

H1. The string experiences oscillations that are small enough so that we can assume that string particles move transversely (perpendicularly) to the equilibrium (rest position) of the taut string.

With this assumption and with coordinates chosen as in the following figure we can describe the oscillations of the string by its transverse displacement $u = u(x,t)$, which is the position at time $t$ of a string particle that is located at position $x$ when it is at rest. We will track string particles with the position vector

$$R = R(x,t) = xi + u(x,t)j$$

as indicated in the foregoing figure. From calculus

$$R_x = i + u_xj$$

is a tangent vector to the string at $R$.

Our model will involve the following relevant variables and forces:
the string in motion has cross sectional areas $A(x,t)$
the string in motion has density $\rho = \rho(x,t)$
the string in motion has length $s = s(x,t)$, measured along the string from its left end at $x = 0$

$T(x,t)$ is the tension (force) exerted by the part of the string to the right of the cross section at $x$ on the cross section at $x$

$F(x,t) = F(x,t)\mathbf{j}$, assumed known, is the impressed transverse force per unit mass that acts along the string

**H2.** The string is perfectly elastic; that is, the tension always acts tangentially.

Fix an instant in time $t$ and consider the part of the string $C_{ab}$ between the cross sections at $a$ and at $b$ with $0 < a < b < L$.

From vector calculus, the differential (infinitesimal) curve length along the string is

$$ds = |d\mathbf{R}| = |\mathbf{R}_x dx| = (1 + u_x^2)^{1/2} dx.$$

Lemma 1, this relation, and conservation of mass give...

$$\tilde{\rho}_0 A_0 = \rho A (1 + u_x^2)^{1/2}. \quad (1)$$

The momentum of the string segment $C_{ab}$ is

$$\int_{C_{ab}} \mathbf{R}_t \rho A \, ds.$$

Newton’s second law, the time rate of change of momentum equals the force, gives

$$\frac{d}{dt} \int_{C_{ab}} \mathbf{R}_t \rho A \, ds = T(b,t) - T(a,t) + \int_{C_{ab}} F(x,t) \rho A \, ds.$$

Use the conservation of mass relation via (1) and the fundamental lemma (Lemma 1) to find

$$\tilde{\rho}_0 A_0 R_{tt} = T_x + \tilde{\rho}_0 A_0 F,$$

$$\tilde{\rho}_0 A_0 u_{tt} \mathbf{j} = T_x + \tilde{\rho}_0 A_0 F \mathbf{j}. \quad (2)$$
Since $\mathbf{R}_x = \mathbf{i} + u_x \mathbf{j}$ is tangent to the string and has a positive $\mathbf{i}$-component, the tension in the string is

$$T = T(x, t) = T(x, t) \frac{\mathbf{R}_x}{|\mathbf{R}_x|} = \frac{T}{(1 + u_x^2)^{1/2}} (\mathbf{i} + u_x \mathbf{j}).$$

(3)

Use (2) and (3) to show that the horizontal component of tension

$$H = \frac{T}{(1 + u_x^2)^{1/2}}$$

is either constant or a function of time.

Combine this observation with (2) and (3) to obtain the inhomogeneous wave equation

$$\tilde{\rho}_0 A_0 u_{tt} = Hu_{xx} + \tilde{\rho}_0 A_0 F.$$

Let

$$\rho_0 = \tilde{\rho}_0 A_0,$$

the linear density of the string at rest and

$$c = \sqrt{\frac{H}{\tilde{\rho}_0}}$$

to express the inhomogeneous wave equation in one spatial dimension as

$$u_{tt} = c^2 u_{xx} + F.$$

If there are no external forces acting on the string, we get the wave equation in one spatial dimension

$$u_{tt} = c^2 u_{xx}.$$

We assume $c$ can be treated as a constant in these wave equations. (However, the derivation shows that $c$ could depend on time. It can also depend on the spatial dimension when the string is inhomogeneous.) Notice that $c$ has units of velocity – it is the velocity of travelling wave solutions of the wave equation.

If the tension in the string is large, the effects of gravity on the vibrations can usually be neglected. However, to model those effects just take $F = g$, the constant gravitational acceleration near the earth. The medium surrounding the string may impede its motion. Such damping is often modeled as proportional to the string’s velocity. In that case, $F = -ku_t$ where $k > 0$ is a constant. If both damping and gravity are modeled, the relevant damped wave equation is

$$u_{tt} + ku_t = c^2 u_{xx} + g.$$

2.2 Wave Motion in Two and Three Spatial Dimensions

Derivations similar to the one given above lead to wave equations in more spatial dimensions. For example, the small transverse oscillations of a two dimensional homogeneous membrane (say a drum head) satisfy

$$u_{tt} = c^2 \Delta u + F$$
which is the inhomogeneous wave equation in two spatial dimensions. When no external forces are acting this reduces to

\[ u_{tt} = c^2 \Delta u \]

which is the wave equation in two spatial dimensions. Add one spatial dimension to these equations and you get the corresponding wave equations in three spatial dimensions.

### 2.3 The Complex Exponential Function and Euler Identities

This section is a brief review of ground covered before and includes a few new observations.

First,

\[
\begin{align*}
e^z &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \quad \text{for all complex } z, \\
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \quad \text{for all complex } z, \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \quad \text{for all complex } z.
\end{align*}
\]

Thus, setting \( z = i\theta \) with \( \theta \) real in the first series and doing some algebra leads to the basic Euler identity

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

This expresses \( e^{i\theta} \) in terms of its real and imaginary parts. Three equivalent forms of this basic identity were given earlier. The same calculations leading to all four identities are valid if \( \theta \) is any complex number. Thus, for any complex number \( z \),

\[
\begin{align*}
e^{iz} &= \cos z + i \sin z, \\
\cos z &= \frac{e^{iz} + e^{-iz}}{2}, \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i}.
\end{align*}
\]

Note that \( e^{iz} = \cos z + i \sin z \) does not express \( e^{iz} \) in terms of its real and imaginary parts when \( z \) in not real.

Second, if \( z = x + iy \), with \( x \) and \( y \) real, then

\[
e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y = e^x \cos y + i e^x \sin y,
\]

which expresses \( \exp(z) \) in terms of its real and imaginary parts. Consequently you can solve the complex equation \( e^z = c \) where \( z = x + iy \) by solving the two real equations

\[
\begin{align*}
e^x \cos y &= \Re(c), \\
e^x \sin y &= \Im(c).
\end{align*}
\]
Easy and useful consequences of the foregoing are:

\[ e^z \neq 0 \text{ for any real or complex number } z. \]
\[ e^z = 1 \iff z = 2\pi in \text{ for } n \text{ any integer.} \]
\[ \cos z \]
\[ \sin z \]
\[ = 0 \text{ or } \pm 1 \text{ have only the real solutions that you already know.} \]

If you take MTH 483/583 you will cover all this and more, putting everything on a firm foundation.

### 2.4 Initial Boundary Value Problems for the Wave Equation

Return to the one dimensional string whose ends are fixed by two posts. Assume the tension is so great that all other forces can be ignored (at least for the time period of interest). If we regard the string as composed of many tiny mass elements, it is clear that we must know the initial position and velocity of each element of the string and how the ends of the string are supported before we can determine the motion of the string. That is, the motion of the string is determined by the initial boundary value problem (IBVP for short)

\[
\begin{align*}
\text{(IBVP)} \quad \left\{ \begin{array}{l}
\ddot{u} = c^2 u_{xx} \quad \text{for } 0 < x < L, \ t > 0, \\
u(x,0) = f(x), \ u_t(x,0) = g(x) \quad \text{for } 0 \leq x \leq L, \\
u(0,t) = 0, \ u(L,t) = 0 \quad \text{for } t \geq 0,
\end{array} \right.
\end{align*}
\]

where \( f(x) \) is the initial displacement and \( g(x) \) is the initial velocity profile of the string.

Compatibility conditions: \( f(0) = f(L) = 0 \) and \( g(0) = g(L) = 0 \).

Let’s try to solve this IBVP by separation of variables ...

Step 1. Since (WE) and (BC) are homogenous linear equations, we start by seeking separated, nontrivial solutions \( u(x,t) = T(t)X(x) \) that satisfy (WE) and (BC).

\( u = TX \) is a solution of (WE) iff (short for “if and only if”) \( T \) and \( X \) are solutions to

\[ \ddot{T} - \lambda c^2 T = 0 \text{ and } X'' - \lambda X = 0, \]

respectively, for any real or complex number \( \lambda \).

\( u = TX, \) with \( T \neq 0, \) will satisfy (WE) and (BC) iff \( X(0) = 0 \) and \( X(L) = 0 \).

**Summary:** \( u = TX \) is a nontrivial solution of (WE) and (BC) iff \( T \) and \( X \) are nontrivial solutions to

\[ \ddot{T} - \lambda c^2 T = 0 \]
and
\[ \begin{cases} 
X'' - \lambda X = 0, \\
X(0) = 0, 
X(L) = 0,
\end{cases} \tag{EVP} \]
respectively, for some real or complex number \( \lambda \).

Important: For any \( \lambda \), \( T \equiv 0 \) and \( X \equiv 0 \) are solutions to the foregoing equations.

We hope there are some choices of \( \lambda \) for which the problem for \( X \) has nontrivial solutions; otherwise, there are no nontrivial separated solutions to (WE) and (BC) and the superposition principle will not enable us to generate further useful solutions. In this context, (EVP) is called an eigenvalue problem. The values of \( \lambda \) (if any) for which (EVP) has nontrivial solutions \( X \) are called eigenvalues. The nontrivial solutions \( X \) are (corresponding) eigenfunctions.

Important: Nonzero multiples of eigenfunctions are eigenfunctions.

Solution of (EVP) yields
\[ \lambda = \lambda_n = -\frac{n^2 \pi^2}{L^2} \text{ for } n = 1, 2, 3, \ldots, \]
(corresponding) eigenfunctions \( X = X_n = \sin \frac{n \pi x}{L} \).

Solve the corresponding \( T \)-equation,
\[ \dddot{T} - \lambda_n c^2 T = 0 \]
for \( T = T_n \) to find separated solutions to (WE) and (BC):
\[ u_n (x, t) = T_n (t) X_n (x) = \left( a_n \cos \frac{n \pi c t}{L} + b_n \sin \frac{n \pi c t}{L} \right) \sin \frac{n \pi x}{L} \]
where \( a_n \) and \( b_n \) are arbitrary constants.

Step 2. Superpose the separated solutions with the goal of finding a solution to (IBVP) of the form
\[ u (x, t) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n \pi c t}{L} + b_n \sin \frac{n \pi c t}{L} \right) \sin \frac{n \pi x}{L}. \tag{4} \]

Let’s see what is required to accomplish this ... :
\[ u (x, 0) = \sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{L} \bigg|_{t=0} = f (x) \quad \text{for } 0 \leq x \leq L, \]
\[ u_t (x, 0) = \sum_{n=1}^{\infty} b_n \frac{n \pi c}{L} \sin \frac{n \pi x}{L} \bigg|_{t=0} = g (x) \quad \text{for } 0 \leq x \leq L. \]
If we are able to determine $a_n$ and $b_n$ so that the foregoing expansions hold, the proposed solution (4) is often called a *formal solution* to the IBVP because an infinite superposition of solutions to WE may no longer satisfy WE (although a finite superposition always does.) This is an important mathematical point with some subtle aspects. We won’t deal with it, but will assume for the problems we are interested in, that a suitable infinite superposition of separated solutions will solve the IBVP of interest.