Physics 651 Fall 2023

Homework #3

(due Wednesday, October 18, 2023)

1. (10 pts) Recall that $Tr(A) = \sum_n A_{nn} = \sum_n \langle \varphi_n | A | \varphi_n \rangle$, where $\{|\varphi_n\rangle\}$ is a complete orthonormal basis. Using bra-ket algebra, prove the following relations:
   (a) Tr (ABC) = Tr (CAB) = Tr (BCA), where A, B, C are operators;
   (b) Tr ($|\psi\rangle\langle\varphi|$) = $\langle\varphi|\psi\rangle$, where $|\varphi\rangle, |\psi\rangle$ are state vectors.

2. (20 pts) Consider matrices $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix}$.
   (a) Are A and B Hermitian? Write down the matrices representing $A^\dagger$ and $B^\dagger$.
   (b) Find eigenvalues and (normalized) eigenvectors of A. What is the relationship between Tr(A) and a sum of the eigenvalues of A? Explain.
   (c) Show that the eigenvectors of A form a (complete and orthonormal) basis.
   (d) Is Tr(AB) = Tr(BA) ? Is det(AB) = det(A)det(B) ? Is det(B^+) = (det(B))*? Show.
   (e) Calculate the commutator [A, B]. Find Tr([A,B]).
   (f) Calculate the inverse of A, i.e. A^{-1}. What are the eigenvalues of A^{-1} ?

3. (15 pts) Consider a system whose Hamiltonian is given by
   $H = \alpha\left(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|\right)$, where α is a real number having the dimensions of energy.
   (a) Is H a projection operator? What about $\alpha^{-2}H^2$?
   (b) Are $|\varphi_i\rangle$ (i=1,2) eigenstates of H?

(c) Assuming that $|\varphi_i>$ (i=1,2) form a complete and orthonormal basis, find the matrix representing H in this basis. What are the eigenvalues and eigenvectors of this matrix?

4. (10 pts) Show that for any two operators A and B,

$$e^{B} A e^{-B} = A + [B, A] + \frac{1}{2!}[B,[B, A]] + \frac{1}{3!}[B,[B,[B, A]]] + ...$$

5. Reading assignment: Sakurai 1.3-1.4.