Solutions of HW #1

Problem #1

(a) $(2,-3,0)$, $(0,0,1)$, $(2i,i,-i)$

$$\left.\begin{aligned} 2a_1 + 2i\,a_3 &= 0 \\ -3a_1 + i\,a_3 &= 0 \\ a_2 - i\,a_3 &= 0 \end{aligned}\right\} \Rightarrow \det\begin{bmatrix} 2 & 0 & 2i \\ -3 & 0 & i \\ 0 & 1 & -i \end{bmatrix} =$$

$$= 2(-i) + 2i(-3) = -8i \neq 0 \Rightarrow \underline{\text{independent}}$$

(b) $(i,1,2)$, $(3,i,-1)$, $(-i,3i,5i)$

$$\det\begin{bmatrix} i & 3 & -i \\ 1 & i & 3i \\ 2 & -1 & 5i \end{bmatrix} = i(-5+3i) - 3(5i-6i) -$$

$$-i(-1-2i) = -5i - 3 - 15i + 18i + i - 2 =$$

$$= -i - 5 \neq 0 \Rightarrow \underline{\text{independent}}$$

(c) $(0,4,0)$, $(i,-3i,i)$, $(2,0,1)$

$$\det\begin{bmatrix} 0 & i & 2 \\ 4 & -3i & 0 \\ 0 & i & 1 \end{bmatrix} = -4(i-2i) = 4i \neq 0 \Rightarrow$$

$$\underline{\text{independent}}$$

problem #2

(a) $2+x^2$, $3-x+4x^3$, $2x+3x^2-8x^3$

$$a_1(2+x^2)+a_2(3-x+4x^3)+a_3(2x+3x^2-8x^3)=0$$

$$x^3(4a_2-8a_3)+x^2(a_1+3a_3)+x(-a_2+2a_3)+$$
$$+2a_1+3a_2=0;$$

$$\left.\begin{array}{l} a_1+3a_3=0 \\ -a_2+2a_3=0 \\ 2a_1+3a_2=0 \end{array}\right| \Rightarrow \det\begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 3 & 0 \end{vmatrix} = -6+3\cdot 2=0$$

$\Downarrow$

dependent

(b) $\sinh^2 x$, 1, $\cosh^2 x$

$$a_1\underbrace{\sinh^2 x}_{\cosh^2 x - 1}+a_2+a_3\cosh^2 x=0$$

$$(a_1+a_3)\cosh^2 x+a_2-a_1=0 \Rightarrow$$

$$a_1=a_2=-a_3\neq 0 \Rightarrow \text{dependent}$$

(c) $x,\ (x-1)^2,\ (x+1)^2$

$$a_1 x + a_2 (x-1)^2 + a_3 (x+1)^2 = 0$$

$$a_1 x + a_2 x^2 - 2a_2 x + a_2 + a_3 x^2 + 2a_3 x + a_3 = 0$$

$$x^2 (a_2 + a_3) + x(a_1 - 2a_2 + 2a_3) +$$

$$+ a_2 + a_3 = 0\ ; \quad a_2 = -a_3,$$

$$a_1 = 2(a_2 - a_3) = 4a_2 \neq 0$$

$\Downarrow$

<u>dependent</u>

(d) $\sin^2 x,\ \cos^2 x,\ \sin 2x$

$$a_1 \sin^2 x + a_2 \cos^2 x + a_3 \sin 2x = 0$$

$$a_1 \frac{1-\cos 2x}{2} + a_2 \frac{1+\cos 2x}{2} + a_3 \sin 2x = 0$$

$$\cos 2x \left(-\frac{a_1}{2} + \frac{a_2}{2}\right) + \sin 2x \cdot a_3 + \frac{a_1 + a_2}{2} = 0$$

$\Downarrow$

$a_3 = 0, \quad a_1 = a_2 = -a_2 = 0 \quad \Rightarrow$ <u>independent</u>

Problem #3

$$|\Psi\rangle = 3i\,|\varphi_1\rangle + |\varphi_2\rangle \; ; \; |\chi\rangle = -\frac{i}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle \; ;$$

(a)

$$\langle\Psi|\Psi\rangle = \left(-3i\,\langle\varphi_1| + \langle\varphi_2|\right)\cdot\left(3i\,|\varphi_1\rangle + |\varphi_2\rangle\right) = 9 + 1 = \underset{\sim}{10}$$

$$\langle\chi|\Psi\rangle = \left(\frac{i}{\sqrt{2}}\langle\varphi_1| + \frac{1}{\sqrt{2}}\langle\varphi_2|\right)\left(3i\,|\varphi_1\rangle + |\varphi_2\rangle\right) = -\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \underset{\sim}{-\frac{2}{\sqrt{2}}}$$

$$\langle\Psi|\chi\rangle = \left(-3i\,\langle\varphi_1| + \langle\varphi_2|\right)\left(-\frac{i}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle\right)$$

$$= -\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \underset{\sim}{-\frac{2}{\sqrt{2}}} = \langle\chi|\Psi\rangle^* \underset{\uparrow}{=} \langle\chi|\Psi\rangle$$

real! (otherwise not!)

$$\langle\chi|\chi\rangle = \left(\frac{i}{\sqrt{2}}\langle\varphi_1| + \frac{1}{\sqrt{2}}\langle\varphi_2|\right)\left(-\frac{i}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle\right)$$

$$= \frac{1}{2} + \frac{1}{2} = \underset{\sim}{1}$$

(B) Problem #3

$$\langle \Psi + \chi | \Psi + \chi \rangle = \left(\left(-3i + \frac{i}{\sqrt{2}}\right) \langle \varphi_1 | + \left(1 + \frac{1}{\sqrt{2}}\right) \langle \varphi_2 | \right) \left(\left(3i - \frac{i}{\sqrt{2}}\right) |\varphi_1\rangle + \left(1 + \frac{1}{\sqrt{2}}\right) |\varphi_2\rangle \right) =$$

$$= \left(3 - \frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2 = 9 - \frac{6}{\sqrt{2}} + \frac{1}{2} + 1 + \frac{2}{\sqrt{2}} + \frac{1}{2} = \underbrace{11 - \frac{4}{\sqrt{2}}}_{} = \underbrace{10}_{\langle\Psi|\Psi\rangle} + \underbrace{1}_{\langle\chi|\chi\rangle} - \underbrace{\frac{2}{\sqrt{2}}}_{\langle\chi|\Psi\rangle} - \underbrace{\frac{2}{\sqrt{2}}}_{\langle\Psi|\chi\rangle}$$

(C) $$|\Psi\rangle\langle\chi| = \left(3i|\varphi_1\rangle + |\varphi_2\rangle\right)\left(\frac{i}{\sqrt{2}}\langle\varphi_1| + \frac{1}{\sqrt{2}}\langle\varphi_2|\right) = -\frac{3}{\sqrt{2}}|\varphi_1\rangle\langle\varphi_1| + \frac{i}{\sqrt{2}}|\varphi_2\rangle\langle\varphi_1| + \frac{3i}{\sqrt{2}}|\varphi_1\rangle\langle\varphi_2| + \frac{1}{\sqrt{2}}|\varphi_2\rangle\langle\varphi_2|$$

$$|\chi\rangle\langle\Psi| = \left(-\frac{i}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle\right)\left(-3i\langle\varphi_1| + \langle\varphi_2|\right) = -\frac{3}{\sqrt{2}}|\varphi_1\rangle\langle\varphi_1| - \frac{3i}{\sqrt{2}}|\varphi_2\rangle\langle\varphi_1| - \frac{i}{\sqrt{2}}|\varphi_1\rangle\langle\varphi_2| + \frac{1}{\sqrt{2}}|\varphi_2\rangle\langle\varphi_2|$$

$$|\chi\rangle\langle\Psi| = \left(|\Psi\rangle\langle\chi|\right)^\dagger \neq |\Psi\rangle\langle\chi|$$

(d) Triangle inequality:

$$\underbrace{\sqrt{\langle \psi+\chi | \psi+\chi \rangle}}_{= 11-\frac{4}{\sqrt{2}}} \leq \sqrt{\underbrace{\langle \psi|\psi\rangle}_{=10}} + \sqrt{\underbrace{\langle \chi|\chi\rangle}_{=1}}$$

from part (a)

$$\sqrt{11-\frac{4}{\sqrt{2}}} \overset{?}{\leq} \sqrt{10}+1 \overset{\text{square}}{\Rightarrow} 11-\frac{4}{\sqrt{2}} \overset{?}{\leq} \underbrace{10+1}_{11}+2\sqrt{10}$$

Obviously $-\frac{4}{\sqrt{2}} < 2\sqrt{10} \Rightarrow$ holds!

(e) Schwarz inequality:

$$|\langle \psi|\chi\rangle|^2 \leq \langle \psi|\psi\rangle\langle \chi|\chi\rangle \Rightarrow$$

from (a) $\Rightarrow \left|\underbrace{-\frac{2}{\sqrt{2}}}_{\langle \psi|\chi\rangle}\right|^2 = 2 \Rightarrow 2 < 10\cdot 1 \Rightarrow$ holds!