QM I

Phys 651

Solutions of HW #2

Problem # 1

(a) $\underbrace{\langle \Psi | A | \varphi \rangle}_{\text{number}} \underbrace{\langle \varphi |}_{\text{bra}} \Rightarrow$ bra

$$\left[\langle \Psi | A | \varphi \rangle \langle \varphi | \right]^{\dagger} = | \varphi \rangle \langle \varphi | A^{\dagger} | \Psi \rangle = \underline{\langle \Psi | A | \varphi \rangle^{*} | \varphi \rangle}$$

(b) $\underbrace{A}_{\text{operators}}\underbrace{|\psi\rangle\langle\psi|}_{}$ ⇒ operator

$$\left[A|\psi\rangle\langle\psi|\right]^{+} = |\psi\rangle\langle\varphi|A^{+} = \underline{|\psi\rangle\langle A\psi|}$$

(c) $\underbrace{\langle\psi|A|\psi\rangle}_{\text{number}}\underbrace{|\psi\rangle\langle\psi|}_{\text{Operator}}\underbrace{A}_{\text{operator}}$ ⇒ Operator

$$\left[\langle\psi|A|\varphi\rangle|\varphi\rangle\langle\psi|A\right]^{+} = A^{+}|\psi\rangle\langle\varphi|\langle\varphi|A^{+}|\psi\rangle$$

$$= \underline{A^{+}|\psi\rangle\langle\varphi|\langle\psi|A|\psi\rangle^{*}}$$

(d) $\underbrace{\langle\varphi|A|\psi\rangle}_{\text{number}}\underbrace{|\varphi\rangle}_{\text{ket}} - i\underbrace{A}_{\text{operator}}\underbrace{|\psi\rangle}_{\text{ket}}$ ⇒ ket

$$\left[\langle\varphi|A|\psi\rangle|\varphi\rangle - iA|\psi\rangle\right]^{+} = \langle\varphi|\langle\psi|A^{+}|\varphi\rangle +$$

$$+i\langle\psi|A^{+} = \underline{\langle\varphi|\langle\varphi|A|\psi\rangle^{*} + i\langle A\psi|}$$

(e) $|\varphi\rangle\langle\varphi|(A+iA)|\psi\rangle\langle\psi|$ ⇒ operator

$$\left[|\varphi\rangle\langle\varphi|(A+iA)|\psi\rangle\langle\psi|\right]^{+} = |\psi\rangle\langle\psi|(-iA^{+}+A^{+})$$

$$|\varphi\rangle\langle\varphi| = \underline{|\psi\rangle\langle\varphi|(+iA+A)|\psi\rangle^{*}\langle\varphi|}$$

Problem # 2

(a) From lecture # 4 $\Rightarrow$ $X^+ = X$ (Hermitian)

$$\left(\frac{d}{dx}\right)^+ = -\frac{d}{dX} \quad \text{(anti-Hermitian)}$$

$$\left(-i\hbar\frac{d}{dx}\right)^+ = -i\hbar\frac{d}{dx} \quad \text{(Hermitian)}$$

From Lecture # 5

$\Downarrow$

$$[F(A)]^+ = F^*(A^+)$$

$\Downarrow$

$$(e^X)^+ = e^{X^+} = e^X \Rightarrow \text{Hermitian}$$

$$\left(e^{d/dX}\right)^+ = e^{(d/dx)^+} = e^{-d/dX} \Rightarrow \text{neither Hermitian nor anti-Hermitian}$$

$$\left(e^{-i\hbar\frac{d}{dx}}\right)^+ = e^{i\hbar\left(\frac{d}{dx}\right)^+} = e^{-i\hbar\frac{d}{dx}} \Rightarrow \text{Hermitian}$$

(b)
$$\left(X\frac{d}{dx}\right)^+ = \underbrace{\left(\frac{d}{dx}\right)^+}_{-d/dx}\underbrace{X^+}_{\overset{\shortparallel}{X}} = -\frac{dX}{dx}$$

Let's act with $\frac{d}{dx}X$ on $\Psi(x)$:

$$\frac{d}{dx}\left(X\Psi(x)\right) = \Psi(x) + X\frac{d}{dx}\Psi(x) =$$

$$= \left(1 + X\frac{d}{dx}\right)\Psi(x) \Rightarrow$$

$$\frac{d}{dx}X = 1 + X\frac{d}{dx} \Rightarrow \left(X\frac{d}{dx}\right)^{+} = -\underset{\uparrow}{1} - X\frac{d}{dx} =$$

identity operator $\equiv I$

$$= XA + B \Rightarrow \begin{aligned} B &= -I \\ A &= -\frac{d}{dx} \end{aligned}$$

Phys. 657
QM I

Problem #3

(a) $A^\dagger = \left[\, i\,(X^2+1)\, d/dx + iX \,\right]^\dagger =$ $\quad \uparrow X^\dagger = X$

$$= -i\left[(X^2+1)\frac{d}{dx}\right]^\dagger - iX =$$

$$= -i\left[\left(\frac{d}{dx}\right)^\dagger X^2 + \left(\frac{d}{dx}\right)^\dagger\right] - iX =$$

$$= -i\left(-\frac{d}{dx}X^2 - \frac{d}{dx}\right) - iX = i\frac{d}{dx}X^2 + i\frac{d}{dx} -$$

$$-iX = i\underline{\frac{d}{dx}X^2} + iX^2\frac{d}{dx} - i\underline{X^2\frac{d}{dx}} + i\frac{d}{dx} - iX$$

$$\textcircled{=}\ \underbrace{i\left[\frac{d}{dx}, X^2\right]}_{=2X} + iX^2\frac{d}{dx} + i\frac{d}{dx} - iX \ \textcircled{=}$$

$\uparrow$

commutator

$\downarrow$

$$\left[\frac{d}{dx}, X^2\right] \underset{\uparrow}{=} \left[\frac{d}{dx}, X\right]X + X\left[\frac{d}{dx}, X\right]$$

$$[A, B^2] = [A, B]B + B[A, B]$$

(distributivity)

$$\left[\frac{d}{dx}, X\right] = ? \Rightarrow \left[\frac{d}{dx}, X\right]\psi = \frac{d}{dx}(X\psi) - X\frac{d\psi}{dx} = \psi + \cancel{X\frac{d\psi}{dx}} - \cancel{X\frac{d\psi}{dx}} \Rightarrow \boxed{\left[\frac{d}{dx}, X\right] = 1}$$

Then, $\left[\frac{d}{dx}, X^2\right] = X + X = 2X$

$$\textcircled{=}\ i\cdot 2X + iX^2\frac{d}{dx} + i\frac{d}{dx} - iX =$$

$$= i(X^2+1)\frac{d}{dx} + iX = A \Rightarrow A^{+} = A$$

(b) $A\psi(x) = 0 \Rightarrow i\cdot(X^2+1)\frac{d\psi(x)}{dx} + iX\psi(x) = 0,$

$$\frac{d\psi(x)}{dx} = -\frac{X}{X^2+1}\psi(x)\ ; \quad \frac{d\psi(x)}{\psi(x)} = -\frac{X}{X^2+1}dx$$

$$\int \frac{d\Psi(x)}{\Psi(x)} = -\int \frac{x\,dx}{x^2+1} \Rightarrow \ln\Psi(x) = -\frac{1}{2}\ln(x^2+1) + \underbrace{\text{const}}_{\text{say, } \ln \underset{\uparrow \atop \text{const}}{B}}$$

$$\underbrace{\Psi(x) = \frac{B \swarrow \text{const}}{\sqrt{x^2+1}}}$$

Normalization : $\displaystyle\int_{-\infty}^{+\infty} |\Psi(x)|^2\,dx = \underset{\sim}{1} =$

$$= B^2 \int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = B^2 \cdot \underbrace{\arctan x \Big|_{-\infty}^{+\infty}}_{\pi} = \underset{\sim}{B^2 \cdot \pi} \Rightarrow$$

$$\Rightarrow \underbrace{B = \frac{1}{\sqrt{\pi}}} \Rightarrow \underbrace{\Psi(x) = \frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{x^2+1}}}$$

(c) $\displaystyle P = \int_{-1}^{1} |\Psi(x)|^2\,dx = \frac{1}{\pi}\int_{-1}^{1}\frac{dx}{x^2+1} =$

$$= \frac{1}{\pi}\underbrace{\arctan x \Big|_{-1}^{1}}_{\frac{\pi}{2}} = \boxed{\frac{1}{2}}$$