Lecture #13

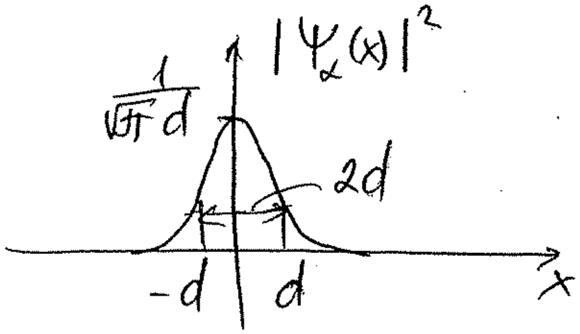
Gaussian wave packets

Consider a Gaussian wave packet whose wave function is given by

$$\langle x | \alpha \rangle = \psi_\alpha(x) = \frac{1}{\pi^{1/4}\sqrt{d}}\, e^{ikx}\, e^{-x^2/(2d^2)}$$

- What is the probability for finding a particle between x and $x+dx$? $\Rightarrow$

$$\mathcal{P}(x)\,dx = |\psi_\alpha(x)|^2 dx = \underbrace{\frac{1}{\sqrt{\pi}\, d}\, e^{-x^2/d^2}}_{\text{prob. density (Gaussian)}} dx$$



- What is the probability for finding a particle with the momentum between p and $p+dp$? $\Rightarrow$

$$\mathcal{P}(p)\,dp = |\phi_\alpha(p)|^2 dp$$

②

$$\Phi_\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{ipx}{\hbar}}\, \Psi_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \cdot \frac{1}{\pi^{1/4}\sqrt{d}} \cdot$$

$$\cdot \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar}px}\, e^{ikx}\, e^{-\frac{x^2}{2d^2}} = \frac{1}{\sqrt{2\pi\hbar}}\, \frac{1}{\pi^{1/4}\sqrt{d}} \cdot$$

$$\cdot \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar}(p-\hbar k)x}\, e^{-\frac{x^2}{2d^2}} \underset{\uparrow}{=} \frac{1}{\sqrt{2\pi\hbar}\,\pi^{1/4}\sqrt{d}} \cdot \int_{-\infty}^{+\infty} dx\, e^{-a^2} \cdot$$

$$-\frac{i}{\hbar}(p-\hbar k)x - \frac{x^2}{2d^2} = -\Big(\underbrace{\frac{x}{\sqrt{2}d} + \frac{i}{\hbar}(p-\hbar k)\frac{d}{\sqrt{2}}}_{=a}\Big)^2 - \frac{(p-\hbar k)^2}{2\hbar^2}d^2$$

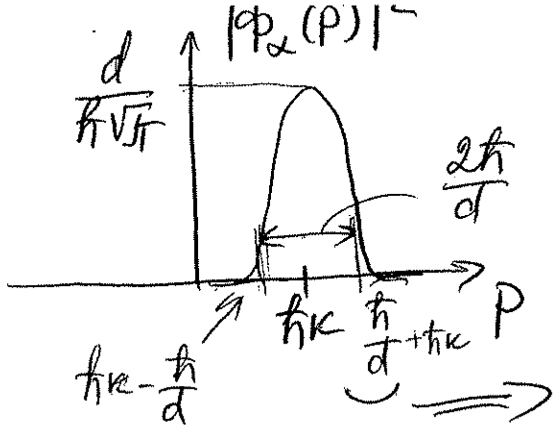
$$\cdot e^{-\frac{(p-\hbar k)^2}{2\hbar^2}d^2} \underset{\uparrow}{=} \frac{\sqrt{2}\,d}{\sqrt{2\pi\hbar d}\,\pi^{1/4}} \underbrace{\int_{-\infty}^{+\infty} da\, e^{-a^2}}_{=\sqrt{\pi}} \cdot e^{-\frac{(p-\hbar k)^2}{2\hbar^2}d^2} =$$

$$da = \frac{dx}{\sqrt{2}d}$$

$$= \frac{d}{\sqrt{\hbar d}\,\pi^{1/4}}\, e^{-\frac{(p-\hbar k)^2}{2\hbar^2}d^2} = \Phi_\alpha(p)$$

So, $\mathcal{P}(p)\,dp = \frac{d}{\hbar\sqrt{\pi}}\, e^{-\frac{(p-\hbar k)^2}{\hbar^2}d^2}\, dp$

probability density is Gaussian



the wider $|\psi_\alpha(x)|^2 \Rightarrow$
the narrower $|\phi_\alpha(p)|^2$

$\Uparrow$

uncertainty principle

- What is the expectation value of x?

$$\langle X \rangle = \int_{-\infty}^{+\infty} dx' \langle \alpha | x' \rangle \langle x' | X | \alpha \rangle \stackrel{}{=}$$

insert $\int dx'' |x''\rangle$ and $\langle x'|X|x''\rangle = \delta(x'$

$$= \int_{-\infty}^{+\infty} \psi_\alpha^*(x')\, x'\, \psi_\alpha(x')\, dx' = \int_{-\infty}^{+\infty} |\psi_\alpha(x')|^2 x' dx' =$$

$$= \frac{1}{\sqrt{\pi}\, d} \underbrace{\int_{-\infty}^{+\infty} e^{-x'^2/d^2} x' dx'}_{\text{odd function}} = 0$$

- $\langle P \rangle = ?$ Let's do it in two ways

1) $\langle P \rangle = \int dx' dx'' \langle \alpha | x' \rangle \underbrace{\langle x' | P | x'' \rangle}_{= -i\hbar \frac{\partial}{\partial x'} \delta(x' - x'')} \langle x'' | \alpha \rangle$

$$= \int_{-\infty}^{+\infty} dx' \, \psi_\alpha^*(x') \left(-i\hbar \frac{\partial \psi_\alpha(x')}{\partial x'}\right) =$$

$$= \int_{-\infty}^{+\infty} dx' \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx'} e^{-x'^2/2d^2} \cdot (-i\hbar) \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx'} \cdot$$

$$\cdot e^{-x'^2/2d^2} \cdot \left(ik - \frac{x'}{d^2}\right) = \frac{1}{\sqrt{\pi}d}(-i\hbar) \int_{-\infty}^{+\infty} e^{-x'^2/d^2}\left(ik - \frac{x'}{d^2}\right) dx'$$

$$= \frac{-i\hbar}{\sqrt{\pi}d} \underbrace{\int_{-\infty}^{+\infty} e^{-x'^2/d^2} dx'}_{\sqrt{\pi}d} \cdot ik = \frac{\hbar k}{\cancel{\sqrt{\pi}d}} \cdot \cancel{\sqrt{\pi}d} = \hbar k$$

$$2) \quad \langle \underline{P} \rangle = \int dp' dp'' \langle \alpha | p' \rangle \overbrace{\langle p' | \underline{P} | p'' \rangle}^{= p'\delta(p'-p'')} \langle p'' | \alpha \rangle$$

$$= \int dp' \, \varphi_\alpha^*(p') \, p' \, \varphi_\alpha(p') = \int_{-\infty}^{+\infty} dp' \, |\varphi_\alpha(p')|^2 p' =$$

$$= \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{+\infty} dp' \, e^{-\frac{(p'-\hbar k)^2}{\hbar^2} d^2} p' = \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{+\infty} dp \, e^{-\frac{p^2 d^2}{\hbar^2}} (p + \hbar k) =$$

change of variables
$p = p' - \hbar k$

$$= \frac{d}{\hbar\sqrt{\pi}} \hbar k \underbrace{\int_{-\infty}^{+\infty} dp \, e^{-\frac{p^2 d^2}{\hbar^2}}}_{= \sqrt{\pi}\frac{\hbar}{d}} = \hbar k$$

Could we do the same with $\langle X \rangle$? $\Rightarrow$ of course!

$$\langle X \rangle = \int_{-\infty}^{+\infty} dp' dp'' \underbrace{\langle \alpha | p' \rangle}_{= \varphi_\alpha^*(p')} \underbrace{\langle p' | X | p'' \rangle}_{\Downarrow} \underbrace{\langle p'' | \alpha \rangle}_{= \varphi_\alpha(p'')}$$

Homework!
(representation of X in $|p\rangle$-basis)

Also, as a part of homework, show that $\langle X^2 \rangle = \dfrac{d^2}{2}$

$$\langle P^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

Then, we can calculate uncertainties $\Rightarrow$

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{d}{\sqrt{2}}$$

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{\frac{\hbar^2}{2d^2} + \hbar^2 k^2 - \hbar^2 k^2} = \frac{\hbar}{\sqrt{2}\, d}$$

The uncertainty relation $\Rightarrow$

$$\Delta X \cdot \Delta P = \frac{d}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2}\, d} = \frac{\hbar}{2} \geq \frac{\hbar}{2} \Rightarrow$$

In the case of a Gaussian wave packet we have <u>equality</u> instead of more general inequality $\Delta X \cdot \Delta P \geq \frac{\hbar}{2}$

that's why a Gaussian wave packet is called a <u>minimum uncertainty packet</u>.

<u>Analysis:</u> If $d \to \infty \implies \Psi_\alpha(x) \sim e^{ikx}$

$\Downarrow$ ⇓ delocalized $\iff$ plane wave extending over entire space

$\varphi_\alpha(p) \to \delta(p - \hbar k)$

$\Downarrow$

localized at $p = \hbar k$

If $d \to 0 \implies \Psi_\alpha(x) \to \delta(x)$ ← localized at $x = 0$

$\varphi_\alpha(p) \to \dfrac{\sqrt{d}}{\sqrt{\hbar}\,\pi^{1/4}}$ ← constant, independent of p

$\Downarrow$

momentum can be anything with equal probability