Lecture # 22

Dirac's operator method

Now let's solve the harmonic oscillator problem in a different way. $H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}$

Define two new operators $\Rightarrow$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right) \leftarrow \text{annihilation operator}$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right) \leftarrow \text{creation operator}$$

a, a^+ are <u>not</u> Hermitian, and $(a)^+ = a^+$

$$[a, a^+] = \cancel{\frac{m\omega}{2\hbar}} \cdot \frac{i}{\cancel{m\omega}}\Big(\underbrace{[P, X]}_{= -i\hbar} - \underbrace{[X, P]}_{= i\hbar}\Big) =$$

$$= \frac{i}{2\hbar}(-2i\hbar) = 1$$

Define the <u>number operator</u> $N = a^+ a$

$\uparrow$ Hermitian

$$a^+ a = \frac{m\omega}{2\hbar}\left(X^2 + \frac{P^2}{m^2\omega^2} - \frac{i}{m\omega}\big(\underbrace{PX - XP}_{= -i\hbar}\big)\right) =$$

$$= \frac{H}{\hbar\omega} - \frac{1}{2} \Rightarrow H = \hbar\omega\left(N + \frac{1}{2}\right)$$

Obviously, $[H, N] = 0 \Rightarrow$ share eigenstates $\Rightarrow$

$$N|n\rangle = n|n\rangle \Rightarrow H|n\rangle = \underbrace{\hbar\omega\left(n + \frac{1}{2}\right)}_{E_n}|n\rangle$$

What is the physical significance of N, a, and a^+ ②

consider

$$[N, a] = [a^+a, a] = a^+\underbrace{[a, a]}_{=0} + \underbrace{[a^+, a]}_{-1}a = -a$$

Similarly, $[N, a^+] = a^+ = Na^+ - a^+N$

Then, $\underbrace{Na^+}_{= a^+ + a^+N}|n> = a^+|n> + a^+\underbrace{N|n>}_{= n|n>} = \underline{(n+1)a^+|n>}$

$\underbrace{N a}_{= -a + aN}|n> = \underline{(n-1)\,a|n>}$

If we define $a|n> \Rightarrow |n-1> \Rightarrow$

$$N|n-1> = (n-1)\,|n-1>$$

$$a^+|n> \Rightarrow |n+1> \Rightarrow$$

$$N|n+1> = (n+1)\,|n+1>$$

So, the action of a is to decrease n by 1

— — — — a^+ increase — — — $\Rightarrow$

which means to decrease / increase the energy by $\hbar\omega$

So, $a|n\rangle = c|n-1\rangle \Rightarrow$

Since both $|n\rangle$ and $|n-1\rangle$ have to be normalized,

$$\langle n|\underbrace{a^+a}_{=N}|n\rangle = |c|^2 = n \Rightarrow \underbrace{c = \sqrt{n}}$$

choose real and positive

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

Similarly, $a^+|n\rangle = \tilde{c}\,|n+1\rangle$

$$|\tilde{c}|^2 = \langle n|\underbrace{aa^+}_{=N+1}|n\rangle = n+1$$

$$\boxed{a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

What happens if we keep applying the annihilation operator a?

$$a^2|n\rangle = \sqrt{n}\,a|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle$$

$$a^3|n\rangle = \sqrt{n}\sqrt{n-1}\sqrt{n-2}\,|n-3\rangle$$

until what limit? can n be negative?

Consider $n = \langle n|N|n\rangle = \langle n|a^+a|n\rangle =$

$$= \underbrace{(\langle n|a^+)(a|n\rangle)}_{\text{norm!}} \geq 0 \Rightarrow$$

(of $a|n\rangle$)

must terminate the sequence at $n=0$,
so that $a|0\rangle = 0$

$\Downarrow$

$E_0 = \frac{\hbar\omega}{2}$

$\uparrow$

ground state energy

What if we apply the creation operator a^+ many times?

$\Downarrow$

start from $|0\rangle$: $a^+|0\rangle = |1\rangle$

$a^+|1\rangle = \sqrt{2}\,|2\rangle$

$a^+|2\rangle = \sqrt{3}\,|3\rangle \Rightarrow$

$|3\rangle = \frac{1}{\sqrt{3}} a^+|2\rangle =$

$= \frac{1}{\sqrt{3}} a^+\left(\frac{a^+}{\sqrt{2}}\right)|1\rangle = \frac{(a^+)^3}{\sqrt{2}\sqrt{3}}|0\rangle \Rightarrow$

generally, $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$

The matrix elements $\langle n'|a|n\rangle = \sqrt{n}\,\delta_{n',n-1}$

$\langle n'|a^+|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

Recall that

$X = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$; $P = i\sqrt{\frac{m\hbar\omega}{2}}(a^+-a)$

$\Downarrow$

Homework: find $\langle n'|X|n\rangle$ and compare to that you found

Using Hermite polynomials!

$$\langle n' | P | n \rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

Note: neither X or P is diagonal(!) in the N-representation, since $[X, H]$ and therefore, $[X, N] \neq 0$ (same for P)

Let's say we know

$|0\rangle, |1\rangle, \ldots |n\rangle$

How do we generate wave functions in the position space? $\Rightarrow$ start from the ground state

$$a|0\rangle = 0$$

$\Rightarrow$ multiply by $\langle x'|$ $\Rightarrow$

$$\langle x'|a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x'| X + \frac{iP}{m\omega}|0\rangle = 0 \Rightarrow$$

$$x'\langle x'|0\rangle + \frac{i}{m\omega}\underbrace{\langle x'|P}_{=\langle x'|\left(-i\hbar\frac{d}{dx}\right)}|0\rangle = 0 \Rightarrow$$

$$\left(x' + \frac{\hbar}{m\omega}\frac{d}{dx'}\right)\langle x'|0\rangle = 0 \Rightarrow$$

diff. equation for the ground-state wave func. $\langle x'|0\rangle$

$$\Downarrow$$

$$\langle x'|0\rangle = \frac{(m\omega)^{1/4}}{\pi^{1/4}\hbar^{1/4}}e^{-\left(x'/\sqrt{\frac{\hbar}{m\omega}}\right)^2/2}$$