Lecture # 23

Uncertainty relations in the case of 1D harmonic oscillator

Consider $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$

$\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$

To evaluate these ⇒ use number representation ⇒

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+) \; ; \; P = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+ - a)$$

$$\langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|\overbrace{a}^{\Rightarrow |n-1\rangle}+\underbrace{a^+|n}_{\Downarrow\, |n+1\rangle}\rangle = 0$$

$\langle P\rangle = 0$ ← similarly

$$\langle X^2\rangle = \frac{\hbar}{2m\omega}\langle n|a^2 + aa^+ + a^+a + a^{+2}|n\rangle =$$

$$= \frac{\hbar}{2m\omega}\langle n|\underbrace{aa^+}_{N+1} + \underbrace{a^+a}_{N}|n\rangle \underset{N|n\rangle = n|n\rangle}{=} \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle P^2\rangle = -\frac{m\hbar\omega}{2}\langle n|a^{+2} + a^2 - \underbrace{a^+a}_{N} - \underbrace{aa^+}_{N+1}|n\rangle =$$

$$= \frac{m\hbar\omega}{2}(2n+1)$$

So, in a general case (arbitrary n) $\Rightarrow$

$$\Delta X = \sqrt{\frac{\hbar}{2m\omega}(2n+1)} \ ; \quad \Delta P = \sqrt{\frac{m\hbar\omega}{2}(2n+1)}$$

$$(\Delta X)^2 \cdot (\Delta P)^2 = \frac{\hbar}{2m\omega}(2n+1) \cdot \frac{m\hbar\omega}{2}(2n+1) =$$

$$= \frac{\hbar^2}{4}(2n+1)^2 \geq \frac{\hbar^2}{4}$$

Recall: for arbitrary $A, B \Rightarrow$

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq \frac{1}{4} |\langle [A,B] \rangle|^2$$

$$A = X, \ B = P \quad [A,B] = i\hbar$$

For ground state $\Rightarrow n = 0 \Rightarrow (\Delta X)^2 (\Delta P)^2 = \frac{\hbar^2}{4}$

$\left(E_0 = \frac{\hbar\omega}{2}\right)$

For a ground state $\Rightarrow$ minimal uncertainty (i.e. inequality turns into equality)

does it make sense?

Recall Lecture #14 $\Rightarrow$ minimal uncertainty is observed in Gaussian wave packets

consider ground state $\psi_n(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}\, 2^n n!}}\, e^{-\alpha^2 x^2/2} H_n(\alpha x)$, $n = 0$

$$\psi_0(x) = \sqrt{\frac{m\omega}{\hbar\sqrt{\pi}}}\, e^{-\frac{m\omega}{2\hbar}x^2} \cdot 1 \qquad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$\rightarrow$ Gaussian! $\Rightarrow \varphi(p)$ will also be Gaussian

$\Downarrow$

minimal uncertainty is expected!

Time development of the oscillator

So far we have not considered time-dependence of our states or operators describing harmonic oscillator. All operators (X, P, a, a^+) were regarded as time-independent (Schrödinger picture) or or time-dependent, but taken at $t=0$ (Heisenberg picture).

Let's work in the Heisenberg picture (recall Lecture #17)

$$\frac{dA_H^{(H)}}{dt} = \frac{1}{i\hbar}\left[A_H, H\right]$$ ⇐ equation of motion for the operator A

$$\frac{dP_H}{dt} = \frac{1}{i\hbar}\left[P_H, \frac{P_H^2}{2m} + \frac{m\omega^2 X_H^2}{2}\right] = \frac{1}{i\hbar}\,\frac{m\omega^2}{2}\cdot$$

$$\cdot \left[P_H, X_H^2\right] = -m\omega^2 X_H$$

$$\underbrace{\left[P_H, X_H\right]}_{=-i\hbar} X_H + X_H \underbrace{\left[P_H, X_H\right]}_{=-i\hbar} = -2i\hbar X_H$$

$$\frac{dX_H}{dt} = \frac{1}{i\hbar}[X_H, H] = \frac{1}{i\hbar}\left[X_H, \frac{P_H^2}{2m}\right] = \frac{1}{2mi\hbar} \cdot$$

$$\cdot \underbrace{[X_H, P_H^2]}_{\parallel} = \frac{P_H}{m}$$

$$\underbrace{[X_H, P_H]}_{=i\hbar} P_H + P_H \underbrace{[X_H, P_H]}_{=i\hbar} = 2i\hbar P_H$$

Recall that $a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{i}{m\omega}P\right)$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{i}{m\omega}P\right)$$

Then, $\frac{da_H}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{dX_H}{dt} + \frac{i}{m\omega}\frac{dP_H}{dt}\right) =$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{P_H}{m} + \frac{i}{m\omega}\cdot(-m\omega^2 X_H)\right) =$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{P_H}{m} - i\omega X_H\right) = -i\omega a_H$$

$$-i\omega a_H = \sqrt{\frac{m\omega}{2\hbar}}\left(-i\omega X_H + \frac{P_H}{m}\right)$$

$$\frac{da_H}{dt} = -i\omega a_H \Rightarrow \boxed{a_H(t) = a_H(0)e^{-i\omega t}}$$

Similarly, $\frac{da_H^+}{dt} = i\omega a_H^+ \Rightarrow$

$$\boxed{a_H^+(t) = a_H^+(0)e^{i\omega t}}$$

What about N_H, H_H ? $\Rightarrow$ (5)

$N_H = a_H^+ a_H = a_H^+(0)\, a_H(0) \Rightarrow$ time-indep.

The same with the Hamiltonian $H = (N + \frac{1}{2})\hbar\omega$

$\Downarrow$

Does it made sense in the Heisenberg picture?

$$\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H]$$

Obviously if $A_H = H$ or $N \Rightarrow \frac{dA_H}{dt} = 0$

$\Downarrow$

in the absence of external forces $\Leftarrow$ consequence of energy conservation!

Since $X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$, $P = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+ - a)$

$\Downarrow$

$$X_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{a_H(0)}_{=\sqrt{\frac{m\omega}{2\hbar}}\left(X_H(0) + \frac{i}{m\omega}P_H(0)\right)} e^{-i\omega t} + a_H^+(0)\, e^{i\omega t}\right) =$$

$$= \frac{1}{2}\left(\left(X_H(0) + \frac{i}{m\omega}P_H(0)\right)e^{-i\omega t} + \left(X_H(0) - \frac{i}{m\omega}P_H(0)\right)\cdot e^{i\omega t}\right) = X_H(0)\cos\omega t + \frac{P_H(0)}{m\omega}\sin\omega t$$

Similarly, $P_H(t) = -m\omega X_H(0)\sin\omega t + P_H(0)\cos\omega t$

Similar to classical equations of motion!

Alternatively, instead of solving the Heisenberg equation of motion, propagate X directly using $\hat{U}(t, \underset{0}{\underset{\|}{t_0}})$ ← for simplicity

$$X_H(t) = \underbrace{\hat{U}^{+}(t,0)}_{= e^{\frac{i}{\hbar}Ht}} X_H(0) \underbrace{\hat{U}(t,0)}_{= e^{-\frac{i}{\hbar}Ht}}$$

Recall HW#5:

$$e^{B} A e^{-B} = A + [B, A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B, [B,[B,A]]] + \ldots$$

Then,

$$X_H(t) = X_H(0) + \left[\frac{i}{\hbar}Ht, X_H(0)\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 [H,[H,X_H(0)]] + \ldots$$

$$= X_H(0) + \frac{it}{\hbar} \cdot \frac{-i\hbar}{m} P_H(0) + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 \cdot \frac{-i\hbar}{m}[H, P_H(0)] + \ldots =$$

$$[H, X_H(0)] = \left[\frac{P_H^2(0)}{2m}, X_H(0)\right] + \left[\frac{m\omega^2 X_H^2(0)}{2}, X_H(0)\right] = \frac{1}{2m}[P_H^2(0), X_H(0)] =$$

$$= \frac{1}{2m}\left\{\underbrace{[P_H(0), X_H(0)]}_{= -i\hbar} P_H(0) + P_H(0)\underbrace{[P_H(0), X_H(0)]}_{= -i\hbar}\right\} = -\frac{i\hbar}{m} P_H(0)$$

$$= X_H(0) + \frac{P_H(0)}{m}t - \frac{1}{2!}t^2\omega^2 X_H(0) + \ldots =$$

$$[H, P_H(0)] = \frac{m\omega^2}{2}[X_H^2(0), P_H(0)] = i\hbar m\omega^2 X_H(0)$$

$= X_H(0)\cos\omega t + \frac{P_H(0)}{m\omega}\sin\omega t$ (7)

So, the operators of position & momentum oscillate in time. What happens to their expectation values? $\Rightarrow$

$$\langle n|X_H(t)|n\rangle = \cos\omega t\,\overbrace{\langle n|X_H(0)|n\rangle}^{=0} +$$

$$+\frac{1}{m\omega}\sin\omega t\,\underbrace{\langle n|P_H(0)|n\rangle}_{=0} = 0 \Rightarrow$$ makes sense, since the expectation values with respect to a stationary state don't change

Recall Lecture #16 $\Rightarrow$

What if we are in some state $|\alpha\rangle = \sum_n c_n|n\rangle$?
(at $t=0$)
and all t's in the Heisenberg picture

$$\langle\alpha|X_H(t)|\alpha\rangle = \cos\omega t\sum_{n,m}\langle n|X_H(0)|m\rangle c_n^* c_m +$$

$$+\frac{1}{m\omega}\sin\omega t\sum_{n,m}\langle n|P_H(0)|m\rangle c_n^* c_m \quad (24.1)$$

$\Rightarrow$ time-dependent expectation value

How is QM oscillator different from CM one? (8) $\Rightarrow$

If you compare time-dependence of $x(t)$ in the case of classical oscillator with that of the $\langle x \rangle (t)$ of QM oscillator $\Rightarrow$ you see that $x(t)$ oscillates,

but $\langle x \rangle (t)$ is <u>time-indep</u> if evaluated with respect to the energy eigenstates $|n\rangle$!!

(equation is similar to that for the QM <u>operator</u> $X(t)$)

In what case are CM & QM oscillators similar? (i.e. QM $\langle x \rangle$ depends on time in a same fashion that $x(t)$) $\Rightarrow$ turns out that it happens if $\langle x \rangle$ is evaluated with respect to the state $|\alpha\rangle$, $a|\alpha\rangle = \alpha|\alpha\rangle$

so-called coherent state

annihilation operator

$\Rightarrow$ eigenstate of a

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \text{ where } \underbrace{|C_n|^2 = \frac{\bar{n}^n}{n!} e^{-\bar{n}}}_{\text{Poisson distribution}}$$

mean value

<u>Morning coffee question</u>:
why do we need to consider a <u>coherent</u> state $|\alpha\rangle$ and not just <u>any</u> <u>superposition</u> of states $|n\rangle$ (e.g. see (24.1)) to get $\langle x \rangle (t)$ similar to CM $x(t)$?