QM
phys 651

Lecture # 7

Measurements. Expectation values

Important: understand the difference between eigenvalues and expectation values.

Example: Stern - Gerlach experiment.

Initially the system is in a $|\psi\rangle$ - state.
The apparatus allows us to measure S_z

spin component along z axis

For $S=\frac{1}{2}$ - systems (e.g. electron)

$\Downarrow$

$2S+1 = 2$ outcomes of the experiment $\Rightarrow$

$$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle \Rightarrow$$ eigenvalues are $\pm\frac{\hbar}{2}$;

"up"
spin "down"

$\Downarrow$ eigenstates are $|\pm\rangle$

These are properties of S_z that are independent of the initial state! However, the probability to measure $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ depends on $|\psi\rangle \Rightarrow$

The expectation value $\langle S_z\rangle = +\frac{\hbar}{2}\cdot p(\frac{\hbar}{2}) - \frac{\hbar}{2} p(-\frac{\hbar}{2})$ depends on $|\psi\rangle$ and can be any real value between $-\frac{\hbar}{2}$ and $+\frac{\hbar}{2}$.

Example

Consider a system whose state is given in term of a complete and orthonormal set $\{|\varphi_n\rangle\}$ =>

$$|\Psi\rangle = \frac{1}{\sqrt{19}}|\varphi_1\rangle + \frac{2}{\sqrt{19}}|\varphi_2\rangle + \sqrt{\frac{2}{19}}|\varphi_3\rangle +$$
$$+ \sqrt{\frac{3}{19}}|\varphi_4\rangle + \sqrt{\frac{5}{19}}|\varphi_5\rangle,$$

where $|\varphi_n\rangle$ are eigenstates of the system's Hamiltonian, $H|\varphi_n\rangle = \underbrace{n\varepsilon_0}_{\text{energy}}|\varphi_n\rangle$, $n = 1, 2, 3, 4, 5$

(a) If the energy is measured → what values can be obtained and with what probabilities?

(b) Find the average energy of this system.

Solution:

(a) Possible values of energy are: $E_n = n\varepsilon_0$, i.e. ε_0, $2\varepsilon_0$, $3\varepsilon_0$, $4\varepsilon_0$ and $5\varepsilon_0$

Probabilities: $\mathcal{P}_n(n\varepsilon_0 = E_n) = \dfrac{|\langle\varphi_n|\Psi\rangle|^2}{\langle\Psi|\Psi\rangle} = ?$

$$\langle\Psi|\Psi\rangle = \frac{1}{19} + \frac{4}{19} + \frac{2}{19} + \frac{3}{19} + \frac{5}{19} = \frac{15}{19}$$

$$\mathcal{P}_1(E_1) = \frac{1/19}{15/19} = \frac{1}{15}$$

Similarly, $\mathcal{P}_2(E_2 = 2\varepsilon_0) = \dfrac{|\langle \varphi_2 | \Psi \rangle|^2}{\langle \Psi | \Psi \rangle} =$

$$= \frac{4/19}{15/19} = \frac{4}{15}$$

$\mathcal{P}_3(E_3 = 3\varepsilon_0) = \dfrac{2}{15}$; $\mathcal{P}_4(E_4 = 4\varepsilon_0) = \dfrac{1}{5}$;

$\mathcal{P}_5(E_5 = 5\varepsilon_0) = \dfrac{1}{3}$.

(b) Average energy $\langle H \rangle = E = \sum_{n=1}^{5} E_n \mathcal{P}_n =$

$$= \frac{1}{15}\cdot\varepsilon_0 + \frac{4}{15}\cdot 2\varepsilon_0 + \frac{2}{15}\cdot 3\varepsilon_0 + \frac{1}{5}\cdot 4\varepsilon_0 + \frac{1}{3}\cdot 5\varepsilon_0 =$$

$$= \varepsilon_0 \left(1 + \frac{4}{5} + \frac{5}{3}\right) = \frac{52}{15}\varepsilon_0$$

Alternatively, one could calculate an expectation value as

$$\underline{E} = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\sum_{n=1}^{5} n \varepsilon_0 a_n^2}{\sum_{n=1}^{5} a_n^2} = \frac{52/19}{15/19}\varepsilon_0 =$$

$$= \frac{52}{15}\varepsilon_0$$

where

$|\Psi\rangle = \sum_{n=1}^{5} a_n |\varphi_1\rangle$,

i.e. $a_1 = \frac{1}{\sqrt{19}}, \ldots a_5 = \sqrt{\frac{5}{19}}$

Measurements. Compatible observables.

Consider complete ~~s~~ orthonormal basis $\{|\varphi_n\rangle\}$, which is an eigenbasis of an operator A

$\Downarrow$

$$A|\varphi_n\rangle = a_n|\varphi_n\rangle, \quad \{|\varphi_n\rangle\} \in \mathcal{E}$$

Then, any $\underbrace{|\psi\rangle}_{\text{initial state}} = \sum_n c_n |\varphi_n\rangle$, $\sum_n |c_n|^2 = 1$ normalization

The probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of A is

$$\mathcal{P}(a_n) = |\langle\varphi_n|\psi\rangle|^2$$

When the measurement is done, the system collapses from $|\psi\rangle \Rightarrow |\varphi_n\rangle$, which is equivalent (mathematically) to applying a projection operator $P_n = |\varphi_n\rangle\langle\varphi_n|$ to $|\psi\rangle$

What if a_n is degenerate, i.e. several orthonormal eigenvectors $|\varphi_n^i\rangle$ correspond to a_n? $\Rightarrow$ $A|\varphi_n^i\rangle = a_n|\varphi_n^i\rangle$,

$$i = 1, 2, \ldots, \underset{\uparrow \text{degeneracy}}{g_n}$$

$\Downarrow$

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |\varphi_n^i\rangle \quad , \quad c_n^i = \langle \varphi_n^i | \psi\rangle$$

In this case, $\mathcal{P}(a_n) = \sum_{i=1}^{g_n} |c_n^i|^2 = \sum_{i=1}^{g_n} |\langle\varphi_n^i|\psi\rangle|^2$

So, in what state does our system end up after a measurement with result a_n? $\Rightarrow$ is it $|\varphi_n^1\rangle$, $|\varphi_n^2\rangle$, ..., their superposition, ...?

$\Downarrow$

the measurement event in this case is equivalent to projecting on a sub-space $\mathcal{E}_n$, which has a dimensionality of g_n $\Rightarrow$ $P_n = \sum_{i=1}^{g_n} |\varphi_n^i\rangle\langle\varphi_n^i|$

The state after the measurement is

$$|\psi_n\rangle = \underbrace{\sum_{i=1}^{g_n} |\varphi_n^i\rangle\langle\varphi_n^i|}_{= P_n}\psi\rangle$$

Is $|\Psi_n\rangle$ normalized? $\Rightarrow$

$$\langle \Psi_n | \Psi_n \rangle = \sum_{i,j=1}^{g_n} \underbrace{\langle \varphi_n^j | \varphi_n^i \rangle}_{=\delta_{ij}} \langle \Psi | \varphi_n^j \rangle \langle \varphi_n^i | \Psi \rangle =$$

$$= \sum_{i=1}^{g_n} |\underbrace{\langle \varphi_n^i | \Psi \rangle}_{=C_n^i}|^2 = \underbrace{\sum_{i=1}^{g_n} |C_n^i|^2}_{} \neq 1$$

$$\uparrow$$
$$\sum_n \sum_{i=1}^{g_n} |C_n^i|^2 = 1$$

To find a <u>normalized</u> state after the measurement

$$|\Psi_n\rangle_{(\text{normalized})} = \frac{|\Psi_n\rangle}{\sqrt{\langle \Psi_n | \Psi_n \rangle}} = \frac{P_n |\Psi\rangle}{\sqrt{\langle P_n \Psi | P_n \Psi \rangle}} =$$

$$= \frac{P_n |\Psi\rangle}{\sqrt{\langle \Psi | \underbrace{P_n^+ P_n}_{=P_n^2 = P_n} | \Psi \rangle}} = \underbrace{\frac{P_n |\Psi\rangle}{\sqrt{\langle \Psi | P_n | \Psi \rangle}}}_{}$$

<u>Note:</u> If you do not know the <u>initial</u> state of the system $\Rightarrow$ you can't determine $|\Psi_n\rangle$

You can state that $|\Psi_n\rangle = \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2}} |\varphi_n^1\rangle +$

say, $g_n = 2$

$+ \frac{\beta}{\sqrt{|\alpha|^2+|\beta|^2}} |\varphi_n^2\rangle$, but you don't know α & β.

Another note: Consider two kets $|\Psi\rangle$ and $|\Psi'\rangle = e^{i\theta}|\Psi\rangle$

θ ← real number

Do they represent the same physical state?

$\Downarrow$

If $\langle\Psi|\Psi\rangle = 1 \Rightarrow \langle\Psi'|\Psi'\rangle = 1$ ✓

Probability predicted for an arbitrary measurement

$$|\langle\varphi_n^i|\Psi'\rangle|^2 = |e^{i\theta}\langle\varphi_n^i|\Psi\rangle|^2 = \underbrace{|\langle\varphi_n^i|\Psi\rangle|^2}_{\text{same}}$$

Similarly, for $|\Psi''\rangle = \alpha e^{i\theta}|\Psi\rangle \Rightarrow$

probability would be $\frac{|\langle\varphi_n^i|\Psi''\rangle|^2}{\langle\Psi''|\Psi''\rangle} = |\langle\varphi_n^i|\Psi\rangle|^2$

$\Downarrow$

Physically, $|\Psi\rangle$, $|\Psi'\rangle$ and $|\Psi''\rangle$ are the same

$\Downarrow$

two proportional state vectors represent the same physical state

Now, what about $|\Psi\rangle = \lambda_1 |\Psi_1\rangle + \lambda_2 |\Psi_2\rangle$
and $|\Psi'\rangle = \lambda_1 e^{i\theta_1} |\Psi_1\rangle + \lambda_2 e^{i\theta_2} |\Psi_2\rangle$

Are $|\Psi\rangle$ and $|\Psi'\rangle$ same states physically?

Probabilities of measurements $\Rightarrow$

$$\frac{|\langle\varphi_n|\Psi\rangle|^2}{\langle\Psi|\Psi\rangle} = \frac{|\lambda_1 \langle\varphi_n|\Psi_1\rangle + \lambda_2 \langle\varphi_n|\Psi_2\rangle|^2}{|\lambda_1|^2 + |\lambda_2|^2} =$$

$$= \frac{|\lambda_1|^2}{|\lambda_1|^2+|\lambda_2|^2} |\langle\varphi_n|\Psi_1\rangle|^2 + \frac{|\lambda_2|^2}{|\lambda_1|^2+|\lambda_2|^2} |\langle\varphi_n|\Psi_2\rangle|^2 +$$

$$+ \frac{2\,\mathrm{Re}\left[\lambda_1 \lambda_2^* \langle\varphi_n|\Psi_1\rangle \langle\varphi_n|\Psi_2\rangle^*\right]}{|\lambda_1|^2 + |\lambda_2|^2} \qquad (8.1)$$

$$\frac{|\langle\varphi_n|\Psi'\rangle|^2}{\langle\Psi'|\Psi'\rangle} = \frac{|\lambda_1 e^{i\theta_1} \langle\varphi_n|\Psi_1\rangle + \lambda_2 e^{i\theta_2} \langle\varphi_n|\Psi_2\rangle|^2}{|\lambda_1|^2 + |\lambda_2|^2}$$

$$= \frac{|\lambda_1|^2}{|\lambda_1|^2+|\lambda_2|^2} |\langle\varphi_n|\Psi_1\rangle|^2 + \frac{|\lambda_2|^2}{|\lambda_1|^2+|\lambda_2|^2} |\langle\varphi_n|\Psi_2\rangle|^2 +$$

$$+ \frac{2\,\mathrm{Re}\left[\lambda_1 \lambda_2^* e^{i(\theta_1-\theta_2)} \langle\varphi_n|\Psi_1\rangle \langle\varphi_n|\Psi_2\rangle^*\right]}{|\lambda_1|^2 + |\lambda_2|^2} \qquad (8.2)$$

Compare (7.1) and (7.2) $\Rightarrow$ they would be the same <u>only</u> if $\theta_2 = \theta_1 + 2\pi k$ (k integer)

$\Downarrow$

$$e^{i(\theta_2-\theta_1)} = e^{i2\pi k} = 1$$

$\Downarrow$

Although global phase factor does <u>not</u> matter, <u>relative phases matter!</u>

What if our measurement involves measuring two observables A & B?

$\Downarrow$

depends on whether they are <u>compatible</u>

$\Downarrow$

Two observables are <u>compatible</u> if their corresponding operators commute $\Rightarrow$ <u>$[A, B] = 0$</u>

Otherwise they are <u>incompatible</u>