QM
Phys 651

Lecture # 8 (Supplement)

1

Example of dealing with measurements of various observables

Consider the following operators on a Hilbert space $\mathcal{H}$:

$$\hat{L}_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} ; \quad \hat{L}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\hat{L}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(1) What are the possible values one can obtain if L_z is measured?

Since $\hat{L}_z$ is diagonal $\Rightarrow$ can read off the eigenvalues from the diagonal : $L_z = 1, 0, -1$

(2) Take the state in which $L_z = 1$. In this state what are $\langle \hat{L}_x \rangle$, $\langle \hat{L}_x^2 \rangle$, and ΔL_x?

First determine eigenvector $|L_z = 1\rangle$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} c_1 \text{ - arbitrary} \\ c_2 = c_3 = 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then, $|\Psi\rangle \equiv |L_z=1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\langle \hat{L}_x \rangle = \langle \Psi | \hat{L}_x | \Psi \rangle = [1\ 0\ 0] \begin{bmatrix} 0 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & 0 \end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \cdot$$

$$= [1\ 0\ 0] \begin{bmatrix}0\\ \frac{1}{\sqrt2}\\0\end{bmatrix} = 0$$

$$\langle \hat{L}_x^2 \rangle = \langle \Psi | \hat{L}_x^2 | \Psi \rangle = [1\ 0\ 0] \begin{bmatrix} 0 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = [1\ 0\ 0] \begin{bmatrix} \frac12 & 0 & \frac12 \\ 0 & 1 & 0 \\ \frac12 & 0 & \frac12 \end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = [1\ 0\ 0] \begin{bmatrix}\frac12\\0\\ \frac12\end{bmatrix} = \frac12$$

The uncertainty $\Delta L_x = \sqrt{\langle \Psi | (\Delta \hat{L}_x)^2 | \Psi \rangle} = \sqrt{\langle \Psi | (\hat{L}_x - \underbrace{\langle \hat{L}_x \rangle}_{=0})^2 | \Psi \rangle} =$

$$= \sqrt{\langle \Psi | L_x^2 | \Psi \rangle} = \frac{1}{\sqrt2}$$

(3) Find the normalized eigenstates and the eigenvalues of $\hat{L}_x$ in $\hat{L}_z$ basis $\Rightarrow$

Eigenvalues: $\det \begin{bmatrix} -\lambda & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & -\lambda & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & -\lambda \end{bmatrix} = 0 \Rightarrow -\lambda\left(\lambda^2 - \frac12\right) - $

$$- \frac{1}{\sqrt2}\left(-\frac{\lambda}{\sqrt2}\right) = 0 \Rightarrow$$

$$\underbrace{\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1}_{\text{possible outcomes for } L_x}$$

Eigenstates : $|L_x=0\rangle \Rightarrow \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow$

$$c_2 = 0$$
$$c_1 + c_3 = 0 \Rightarrow |L_x=0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$|L_x=1\rangle \Rightarrow \begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -c_1 + \frac{1}{\sqrt{2}} c_2 = 0 \\ \frac{1}{\sqrt{2}} c_2 - c_3 = 0 \end{array}$

$$|L_x=1\rangle = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \cdot \frac{1}{2}$$

$|L_x=-1\rangle \Rightarrow \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} c_1 + \frac{1}{\sqrt{2}} c_2 = 0 \\ \frac{c_2}{\sqrt{2}} + c_3 = 0 \end{array}$

$$|L_x=-1\rangle = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \cdot \frac{1}{2}$$

(4) If the particle is in the state with $L_z=-1$, and L_x is measured, what are the possible outcomes and their probabilities? $\Rightarrow$ from above, we know that the outcomes are $L_x = 0, \pm 1$

Probabilities : $\mathcal{P}_{L_x=0} = |\langle L_z=-1 | L_x=0\rangle|^2 =$

$$= \left| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{L_x=1} = \left| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$\mathcal{P}_{L_x=-1} = \frac{1}{4}$$

Check: $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

(5) Consider the state $|\Psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$ in the L_z basis.

If L_z^2 is measured in this state and a result $+1$ is obtained, what is the state after the measurement? How probable was this result? If L_z is measured, what are the outcomes and respective probabilities?

$\Downarrow$

Let's find representation of $\hat{L}_z^2$ in the $\hat{L}_z$ basis $\Rightarrow$

$$\hat{L}_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_{1,2} = 1$, $\lambda_3 = 0 \Leftarrow L_z^2$

$\Downarrow$ degenerate eigenvalues!

Eigenvectors: $|L_z^2 = 0\rangle \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow$

$c_1 = 0 = c_3$, c_2 – arbitrary $\Rightarrow |L_z^2 = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$|L_z^2=1\rangle \Rightarrow \begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=0 \Rightarrow c_2=0$$

c_1, c_3 – arbitrary

$$|L_z^2=1;1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \;;\; |L_z^2=1;2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

vector #1 in the $L_z^2=1$ subspace

vector #2 in the $L_z^2=1$ subspace

So, after the measurement we obtained $L_z^2=1$. What state are we in now? $|L_z^2=1;1\rangle$, $|L_z^2=1;2\rangle$ or their combination?

need to find a projection of the initial state on a subspace $L_z^2=1$ (instead of just a projection on the corresponding eigenvector, e.g. $|L_z^2=0\rangle$) $\Rightarrow$

projector

$$|\Psi'\rangle = \frac{P_{L_z^2=1}|\Psi\rangle}{\sqrt{\langle P_{L_z^2=1}\Psi|P_{L_z^2=1}\Psi\rangle}} \;,\quad P_{L_z^2=1} = |L_z^2=1;1\rangle\langle L_z^2=1;1| + |L_z^2=1;2\rangle\langle L_z^2=1;2|$$

state after the measurement

$$P_{L_z^2=1} = \begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&0&0\\\frac{1}{2}&0&\frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2}&0&-\frac{1}{2}\\0&0&0\\-\frac{1}{2}&0&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} =$$

alternatively

$$= I - P_{L_z^2=0} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

(since $\sum_n P_n = 1$!)

Let's find $|\Psi'\rangle \Rightarrow$

$$P_{L_z^2=1}|\Psi\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\langle P_{L_z^2=1}\Psi | P_{L_z^2=1}\Psi\rangle = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{3}{4}$$

So, $$|\Psi'\rangle = \frac{2}{\sqrt{3}}\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

The probability to find the system in the state $|\Psi'\rangle$:

$$\mathcal{P}_{L_z^2=1} = |\langle\Psi'|\Psi\rangle|^2 = \left|\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}\right|^2 = \frac{3}{4}$$

Alternatively, $$\mathcal{P}_{L_z^2=1} = 1 - \mathcal{P}_{L_z^2=0} = 1 - \left|\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}\right|^2 =$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

What if we measure L_z? $\Rightarrow$

$$\mathcal{P}_{L_z=0} = \left|\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{\langle L_z=0|}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}\right|^2 = \frac{1}{4}$$

$$\mathcal{P}_{L_z=1} = \left|\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}\right|^2 = \frac{1}{4}$$

$$\mathcal{P}_{L_z=-1} = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

(6) A particle is in a state for which the probabilities are $\mathcal{P}_{L_z=1} = \frac{1}{4}$, $\mathcal{P}_{L_z=0} = \frac{1}{2}$, $\mathcal{P}_{L_z=-1} = \frac{1}{4}$

The most general, normalized state with this property is

$$|\Psi\rangle = \frac{e^{i\delta_1}}{2}|L_z=1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z=0\rangle + \frac{e^{i\delta_3}}{2}|L_z=-1\rangle$$

Note: if $|\Psi\rangle$ is a normalized state, then the state $e^{i\theta}|\Psi\rangle$ is a physically equivalent normalized state

Does this mean that the factors $e^{i\delta_j}$ $(j=1,2,3)$ are irrelevant?

$\Downarrow$

Let's calculate, for example, probability of measuring $L_x=0$ in a state $|\Psi\rangle$:

$$|L_x=0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad \text{(from above)}$$

$\Downarrow$

$$\mathcal{P}_{L_x=0} = |\langle L_x=0|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}e^{i\delta_1}/2\\ e^{i\delta_2}/\sqrt{2}\\ e^{i\delta_3}/2\end{bmatrix}\right|^2 =$$

$$= \left|\frac{1}{2\sqrt{2}}\left(e^{i\delta_1}-e^{i\delta_3}\right)\right|^2 = \frac{1}{8}\left|e^{i\delta_1}\left(1-e^{i(\delta_3-\delta_1)}\right)\right|^2 = \frac{1}{8}\left(2-2\cos(\delta_3\right.$$

$= \frac{1}{4}(1-\cos(\delta_3-\delta_1))$ ← the probability depends on the

difference between the phases, so even though (8)
<u>absolute</u> phases are irrelevant, the <u>relative</u> ones are!