Physics 652 Winter 2024

Homework #5

(due Wednesday, February 21, 2024)

1. (10 pts) Consider a spin ½ particle.

   (a) Find the eigenvalues (s_n) and eigenstates of the spin operator **S** in the direction of an arbitrary unit vector **n**.

   (b) What is the expectation value of **(S·n)** in the state $|s_n = \hbar/2>$? What about the expectation value of **(S·n)** in the state $|s_z = \hbar/2>$?

2. (10 pts) Find the energy levels of a spin 3/2 particle whose Hamiltonian is given by

   $H = \frac{a}{\hbar^2}\left(S_x^2 + S_y^2 - 2S_z^2\right) - \frac{b}{\hbar}S_z$, where a and b are constants.

3. (15 pts) Sakurai 3.38.

4. (20 pts) Consider the hydrogen atom. So far, we have been taking into account only the Coulomb interaction between the electron and the proton, i.e. the Hamiltonian was $H_{Coulomb}=\mathbf{P}^2/2m - e^2/r$. Now let's take into account the *hyperfine interaction*. The Hamiltonian describing this interaction, which is due to the magnetic moments of the two particles is:

$$H_{hf} = A\mathbf{S_1}\cdot\mathbf{S_2} \qquad (A>0) \qquad (1)$$

(This formula assumes that the orbital state of the electron is $\left|1,0,0\right\rangle$.) The total Hamiltonian is thus $H_{Coulomb} + H_{hf}$. .

(a) Show that H_{hf} splits the ground state into two levels and find these energies (of the triplet and singlet states).

(b) Assume that the electron and proton are two dipoles μ_e and μ_p separated by a distance a_0, with interaction energy of the order $H \cong \frac{\mu_e \mu_p}{a_0^3}$. Then, it can be shown that the constant A of Eq.(1) is $A \sim \frac{e}{mc} \frac{5.6e}{2Mc} \frac{1}{a_0^3}$, where m, M are the masses of the electron and proton, respectively, a_0 is the Bohr radius, e is the electron charge, and c is the speed of light.

Find the energy difference between the triplet and singlet levels $\Delta E = E_{triplet} - E_{singlet}$ in terms of A and show that ΔE is a correction of order $(m/M)\alpha^2$ relative to the ground-state energy, where α is the fine-structure constant ($\alpha = \frac{e^2}{\hbar c}$).

(c) Using the ΔE from part (b), estimate the frequency of the emitted radiation as the atom jumps from the triplet to the singlet. For your information, measurement of this radiation, called 21-cm line, is a way to detect hydrogen in other parts of the universe.

(d) Let's say you have randomly selected 20 hydrogen atoms. Under the condition of thermal equilibrium at room temperature, how many of them will be in the singlet state and how many in the triplet state? Why?

5. Reading assignment: 3.1-3.3, 3.5-3.9 of Sakurai; if you are looking for extra depth on rotational symmetries, see 4.1 as well !