Problem #1

$$L_\pm = L_x \pm i L_y$$

(a) $[L_z, L_\pm] = [L_z, L_x \pm i L_y] = \underbrace{[L_z, L_x]}_{= i\hbar L_y} \pm$

$\pm i \underbrace{[L_z, L_y]}_{= -i\hbar L_x} = i\hbar L_y \pm \hbar L_x =$

$= \pm\hbar \underbrace{(L_x \pm i L_y)}_{= L_\pm} = \pm \hbar L_\pm$

(b) $[\vec{L}^2, L_\pm] = [L_x^2 + L_y^2 + L_z^2, L_x \pm i L_y] =$

$= [L_y^2, L_x] + [L_z^2, L_x] \pm i [L_x^2, L_y] \pm i [L_z^2, L_y]$

$= \cancel{L_y \cdot (-i\hbar) L_z} - \cancel{i\hbar L_z L_y} + i\hbar L_z L_y + \cancel{i\hbar L_y L_z} \pm i \cdot$

$\cdot (L_x \cdot i\hbar L_z + L_z \cdot i\hbar L_x) \pm i \cdot (L_z \cdot (-i\hbar) L_x - i\hbar L_x L_z$

$= 0$

(c) $L_{\mp}L_{\pm} = (L_x \mp i L_y)(L_x \pm i L_y) =$

$$= \underbrace{L_x^2 + L_y^2}_{\vec{L}^2 - L_z^2} \pm i L_x L_y \mp i L_y L_x = \vec{L}^2 - L_z^2 \pm$$

$$\pm i \underbrace{(L_x L_y - L_y L_x)}_{[L_x, L_y] = i\hbar L_z} = \vec{L}^2 - L_z^2 \mp \hbar L_z$$

Problem #2

One of the definitions of $Y_\ell^m(\theta, \varphi) \Rightarrow$

$$Y_\ell^m(\theta,\varphi) = \frac{(-1)^\ell}{2^\ell \ell!} \sqrt{\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}}\, e^{im\varphi} (\sin\theta)^{-m} \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}} (\sin\theta)^{2\ell}$$

Then $\underset{\substack{\uparrow \\ \text{parity operator}}}{\mathcal{P}} Y_\ell^m(\theta,\varphi) = Y_\ell^m(\pi-\theta, \pi+\varphi) = \underset{\substack{\uparrow \\ \text{we need to find this!}}}{a} Y_\ell^m(\theta,\varphi)$

$\varphi \to \pi + \varphi$: $e^{im\varphi} \to e^{im\varphi} \cdot e^{i\pi m} = (-1)^m e^{im\varphi}$

$\theta \to \pi - \theta \Rightarrow \sin\theta \Rightarrow \sin(\pi-\theta) = \sin\theta$

$\cos\theta \Rightarrow \cos(\pi - \theta) = -\cos\theta \Big\} \Rightarrow$

$(\cos\theta)^{\ell-m} \Rightarrow \cos\theta \cdot (-1)^{\ell-m}$

Then, $Y_\ell^m(\pi-\theta, \pi+\varphi) = Y_\ell^m(\theta,\varphi)(-1)^{\ell-m}(-1)^m = (-1)^\ell Y_\ell^m(\theta,\varphi)$

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Problem #3 Sakurai 3.16

$$L_\pm = L_x \pm i L_y \Rightarrow L_x = \frac{L_+ + L_-}{2};$$

$$L_y = \frac{L_+ - L_-}{2i}$$

$$\langle L_x \rangle = \frac{1}{2} \langle l,m | \underbrace{L_+ | l,m \rangle}_{C_+ |l,m+1\rangle} + \frac{1}{2} \langle l,m | \underbrace{L_- | l,m \rangle}_{C_- |l,m-1\rangle}$$

$$= 0$$

$$\langle L_y \rangle = \frac{1}{2i} \left(\langle l,m | L_+ | l,m \rangle - \langle l,m | L_- | l,m \rangle \right) = 0$$

$$\langle L_x^2 \rangle + \langle L_y^2 \rangle = \langle \vec{L}^2 \rangle - \langle L_z^2 \rangle = \hbar^2 l(l+1) - \hbar^2 m^2;$$

$$\langle L_x^2 \rangle = \frac{1}{4} \langle L_+^2 \rangle + \frac{1}{4} \langle L_-^2 \rangle + \frac{1}{4} \langle L_+ L_- \rangle + \frac{1}{4} \langle L_- L_+ \rangle$$

$$= \frac{1}{4} \left(\langle L_+ L_- \rangle + \langle L_- L_+ \rangle \right);$$

$$\langle L_y^2 \rangle = -\frac{1}{4} \left(\langle L_+^2 \rangle + \langle L_-^2 \rangle - \langle L_+ L_- \rangle - \langle L_- L_+ \rangle \right) =$$

$$= \langle L_x^2 \rangle = \frac{\hbar^2 l(l+1) - \hbar^2 m^2}{2};$$

semiclassical interpretation — see Lecture #3