QM II
Phys 652

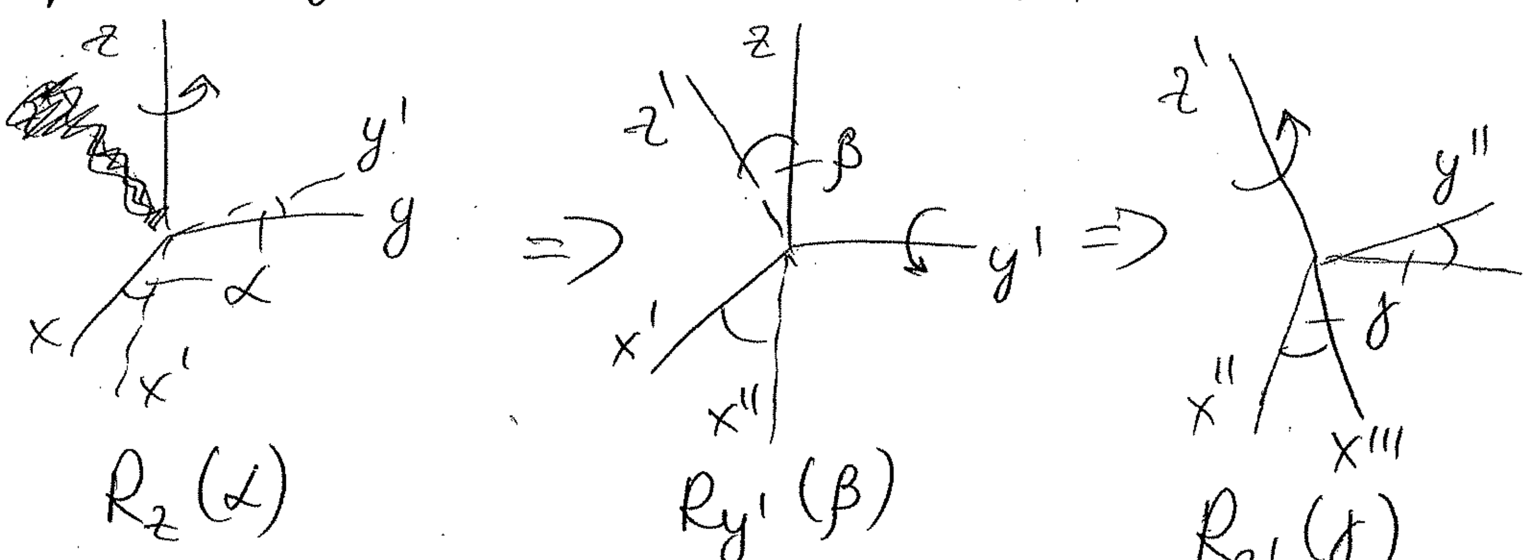
Lecture #11

①

Representations of the rotation operator

Recall Classical Mechanics:

an arbitrary rotation of a rigid body can be accomplished in three steps $\Rightarrow$ Euler rotations, specified by 3 Euler angles $\alpha, \beta, \gamma \Rightarrow$



Total $R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha) =$

$$= R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$$

see Sakurai and Goldstein for details

Now the same in the spin space $\Rightarrow$ for $s = \frac{1}{2}$ $\Rightarrow$

$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\,\mathcal{D}_y(\beta)\,\mathcal{D}_z(\gamma) \doteq$$

$$\doteq e^{-\frac{i}{2}\sigma_z\alpha}\, e^{-\frac{i}{2}\sigma_y\beta}\, e^{-\frac{i\sigma_z}{2}\gamma} = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \cdot$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad ; \; e^{-\frac{i}{2}\varphi\vec{\sigma}\cdot\vec{n}} =$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad = I\cos\frac{\varphi}{2} - $$

$$- i(\vec{\sigma}\cdot\vec{n})\sin\frac{\varphi}{2}$$

$$\cdot \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}}\cos\beta & -e^{i\frac{\gamma}{2}}\sin\frac{\beta}{2} \\ e^{-i\frac{\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} e^{-i\frac{(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}}_{} \qquad (12.1)$$

$j = \frac{1}{2}$ irreducible representation of the rotation operator $\mathcal{D}(\alpha,\beta,\gamma)$

Matrix elements: $\mathcal{D}^{(1/2)}_{m_s' m_s}(\alpha,\beta,\gamma) = \langle \frac{1}{2}, m_s' | e^{-\frac{iS_z\alpha}{\hbar}} \cdot$

$e^{-\frac{iS_y\beta}{\hbar}} e^{-\frac{iS_z\alpha}{\hbar}} |\frac{1}{2}, m_s\rangle$

Now let's generalize it to the case of arbitrary angular momentum J =>

$$\mathcal{D}^{(j)}_{m'm}(R) = \langle j, m' | e^{-i\frac{\vec{J}\cdot\vec{n}}{\hbar}\varphi} | j, m\rangle$$

Wigner functions

why not $\langle j', m' | \dots | j, m\rangle$?

Since $[\vec{J}^2, \vec{J}] = 0 \Rightarrow \vec{J}^2 \underbrace{\mathcal{D}(R)}_{e^{-i\frac{\vec{J}\cdot\vec{n}}{\hbar}\varphi}} = \mathcal{D}(R)\vec{J}^2$

Then, $\vec{J}^2 \mathcal{D}(R)|j, m\rangle =$

$= \mathcal{D}(R)\vec{J}^2|j,m\rangle = \mathcal{D}(R)\hbar^2 j(j+1)|j,m\rangle =$

$= \hbar^2 j(j+1) \mathcal{D}(R)|j, m\rangle$

call it $|j', m\rangle$ => then

So, rotations do not change j! $\Leftarrow j = j' \Leftarrow$ $\left\{ \begin{array}{l} \vec{J}^2|j',m\rangle = \hbar^2 j(j+1)|j',m\rangle \\ \Updownarrow \text{ compare} \\ \vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \end{array} \right.$

$D^{(j)}_{m'm}(R)$ matrix is a $\underbrace{(2j+1)}_{\substack{\uparrow \\ \text{span of} \\ m'}} \times \underbrace{(2j+1)}_{\substack{\uparrow \\ \text{span of} \\ m}}$ matrix $\Rightarrow$

$(2j+1)$-dimensional irreducible representation of the rotation operator $\mathcal{D}(R)$

$\downarrow$

The space $\mathcal{E}_J$ is irreducible with respect to rotations ~~if any arbitrary vector $|j,m\rangle$ can be rotated~~ as $\mathcal{D}^{(j)}(R)\,|j,m\rangle$ to yield another vector $|j,m'\rangle$ and the set of vectors obtained by rotation span the entire $\mathcal{E}_J$. If there existed at least one vector $|j,m''\rangle$ such that $\mathcal{D}^{(j)}(R)\,|j,m''\rangle$ spans only a part of $\mathcal{E}_J$ $\Rightarrow$ $\mathcal{E}_J$ is reducible

Irreducible representation $\Rightarrow$

$$\underset{\substack{\uparrow \\ \text{diagonalized} \\ \text{and can't be broken} \\ \text{into smaller blocks}}}{\mathcal{D}(R)} \doteq \begin{pmatrix} \boxed{\mathcal{D}^{(j_1)}(R)} \leftarrow (2j_1+1)\times(2j_1+1) & & & 0 \\ & \boxed{\mathcal{D}^{(j_2)}(R)} \leftarrow (2j_2+1)\times(2j_2+1) & & \\ & & \ddots & \\ 0 & & (2j_n+1)(2j_n+1) \rightarrow & \boxed{\mathcal{D}^{(j_n)}(R)} \end{pmatrix}$$

How do we find $d^{(j)}_{m'm}(\beta)$ for the case of, say, $j=1$? $\Rightarrow$

1) Recall that $J_y = \frac{J_+ - J_-}{2i}$

2) Find a matrix representation of J_y in $|j,m\rangle$ basis $\Rightarrow$ $j=1 \Rightarrow m=-1,0,1$

$$J_y^{(j=1)} \doteq \frac{\hbar}{2} \begin{array}{c} \begin{array}{ccc} m=1 & 0 & -1 \end{array} \\ \begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix} \end{array} \begin{array}{c} m'= \\ 1 \\ 0 \\ -1 \end{array}$$

$\langle 1, m' | \underbrace{J_y}_{\frac{J_+ - J_-}{2i}} | 1, m\rangle$ $\Rightarrow$ $\underbrace{0 \text{ if } m'=m}$ or if $m'-m=\pm 2$

$\langle 1,1| \frac{J_+ - J_-}{2i} |1,0\rangle = \frac{1}{2i}\hbar \cdot \sqrt{1\cdot 2-0} = \underbrace{\frac{\hbar}{2}\cdot(-i\sqrt{2})}$

$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\, |j, m\pm 1\rangle$

3) Expand $e^{-\frac{i}{\hbar}J_y^{(1)}\beta} = 1 - \frac{i}{\hbar}J_y\beta + \frac{1}{2!}\left(\frac{-i}{\hbar}J_y^{(1)}\beta\right)^2 + \ldots =$

$\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$= 1 - \left(\frac{J_y^{(1)}}{\hbar}\right)^2 (1-\cos\beta) - i\left(\frac{J_y^{(1)}}{\hbar}\right)\sin\beta \Rightarrow$

show! $\left(\frac{J_y^{(1)}}{\hbar}\right)^3 = \frac{J_y^{(1)}}{\hbar}$

$$\Rightarrow d^{(1)}(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ (1-\cos\beta)/2 & (\sin\beta)/\sqrt{2} & \frac{1+\cos\beta}{2} \end{pmatrix}$$

Let's rotate the state $|j, m\rangle \Rightarrow$

$$\mathcal{D}(R)|j, m\rangle = \sum_{m'} |j, m'\rangle\langle j, m'|\mathcal{D}(R)|j, m\rangle$$

m' over $2j+1$ values

$$= \sum_{m'} |j, m'\rangle \underbrace{\mathcal{D}^{(j)}_{m'm}(R)}$$

related to the probability to be found in the state $|j, m'\rangle$

Let's generalise the expression (12.1) obtained for $\mathcal{D}^{(1/2)}_{m_s' m_s}(\alpha, \beta, \gamma)$ to the case of an arbitrary j

$$\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle j, m'| e^{-\frac{i}{\hbar}J_z\alpha} e^{-\frac{i J_y}{\hbar}\beta} e^{-\frac{i J_z \gamma}{\hbar}} |j, m\rangle$$

$\Downarrow$ $e^{-im'\alpha}\langle j, m'|$ $\Downarrow$ $e^{-im\gamma}|j, m\rangle$

$$= e^{-i(m'\alpha + m\gamma)} \cdot \underbrace{\langle j, m'| e^{-\frac{i}{\hbar}J_y\beta}|j, m\rangle}_{\equiv \, d^{(j)}_{m'm}(\beta)}$$

For $j = \frac{1}{2} \Rightarrow d^{(1/2)}_{m'm}(\beta) = \langle j = \frac{1}{2}, m'| e^{-\frac{i}{2}\sigma_y \beta}| \frac{1}{2}, m\rangle$

$$= \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$$ (see p. 2)

Note: there is a general expression for

$d^{(j)}_{m'm}(\beta)$ (Wigner formula) $\Rightarrow$ Wigner, "Group theory"

$$\Downarrow$$

$$d^{(j)}_{m'm}(\beta) = \sum_{k} (-1)^{k+m'-m} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j-m'-k)!\,(j+m-k)!\,(k+m'-m)!\,k!}$$

$$\cdot \left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2k} \left(\sin\frac{\beta}{2}\right)^{m'-m+2k}$$

Sakurai p. 223

summation over k at which the factorials in the denominator are > 0

Recall: $\mathcal{D}(\alpha,\beta,\gamma)\,|j,m\rangle = \sum\limits_{m'=-j}^{j} \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)\,|j,m'\rangle =$

$$= \sum_{m'=-j}^{j} e^{-i(m'\alpha+m\gamma)}\, \underbrace{d^{(j)}_{m'm}(\beta)}\, |j,m'\rangle$$

determines probability to end up in $|j,m'\rangle$ after rotation

Properties of $\mathcal{D}(\alpha,\beta,\gamma)$:

$$\mathcal{D}^{+}(\alpha,\beta,\gamma) = \left[\mathcal{D}_z(\alpha)\,\mathcal{D}_y(\beta)\,\mathcal{D}_z(\gamma)\right]^{+} =$$

$$= \mathcal{D}_z^{+}(\gamma)\,\mathcal{D}_y^{+}(\beta)\,\mathcal{D}_z^{+}(\alpha) = e^{\frac{i}{\hbar}J_z\gamma}\, e^{\frac{i}{\hbar}J_y\beta}\, e^{\frac{i}{\hbar}J_z\alpha} =$$

$$= \mathcal{D}_z(-\gamma)\,\mathcal{D}_y(-\beta)\,\mathcal{D}_z(-\alpha)$$

- $\mathcal{D}^{-1}(\alpha,\beta,\gamma) = \mathcal{D}^{+}(\alpha,\beta,\gamma)$

Properties of $\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)$ and $d^{(j)}_{m'm}(\beta)$.

- $\left[\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)\right]^{*} = \mathcal{D}^{(j)}_{mm'}(-\gamma,-\beta,-\alpha)$

- $\sum_{m'}\left[\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)\right]^{*}\mathcal{D}^{(j)}_{m'k}(\alpha,\beta,\gamma) =$

$$= \sum_{m'} \langle j,m|\underbrace{\mathcal{D}^{+}(\alpha,\beta,\gamma)}_{=\mathcal{D}^{-1}(\alpha,\beta,\gamma)}|j,m'\rangle\langle j,m'| \mathcal{D}(\alpha,\beta,\gamma)|j,k\rangle = \langle j,m|\underbrace{\mathcal{D}^{-1}\mathcal{D}}_{=I}|j,k\rangle =$$

$$= \delta_{mk}$$

- $d^{(j)}_{m'm}(0) = \delta_{m'm}$
- $d^{(j)}_{m'm}(\pi) = (-1)^{j-m}\delta_{m',-m}$ ← morning coffee exercise!!
- Since $d^{(j)}_{m'm}$ are real and represent rotations $\Rightarrow$ unitary + real $\Rightarrow$ orthogonal $d^{(j)}(\beta)$

$$d^{(j)}_{m'm}(\beta) = \left(d^{(j)}_{mm'}(\beta)\right)^{-1} = d^{(j)}_{mm'}(-\beta)$$