Lecture #12

Spherical harmonics as rotation matrices

Consider $Y_\ell^m(\theta,\varphi) = \langle \vec{n} | \ell, m \rangle$ (Sakurai 3.6)

$\uparrow$ direction eigenket

If $|\vec{n}'\rangle = \mathcal{D}(R)|\vec{n}\rangle \Rightarrow Y_\ell^m(\theta',\varphi') = \langle \vec{n}'|\ell, m\rangle$

$\uparrow$ rotated eigenket

How are $Y_\ell^m(\theta,\varphi)$ and $Y_\ell^m(\theta',\varphi')$ related?

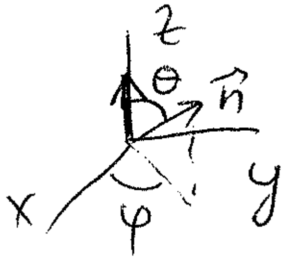
$$|\vec{n}'\rangle = \sum_{m',\ell} |\overset{\ell}{j}, m'\rangle\langle \overset{\ell}{j}, m'|\mathcal{D}(R)|\vec{n}\rangle =$$

$$= \sum_{m'm''\ell} |\ell, m'\rangle \underbrace{\langle \ell, m'|\mathcal{D}(R)|\ell, m''\rangle}_{\mathcal{D}^{(\ell)}_{m'm''}} \underbrace{\langle \ell, m''|\vec{n}\rangle}_{Y_\ell^{m''*}(\theta,\varphi)}$$

Then,

$$Y_\ell^{m*}(\theta',\varphi') = \langle \ell, m|\vec{n}'\rangle = \sum_{m''} \mathcal{D}^{(\ell)}_{mm''} Y_\ell^{m''*}(\theta,\varphi)$$

Let's now approach from a different end: ②



$|\vec{z}\rangle$ in coordinate space

$\Downarrow$

apply $\mathcal{D}(R)$ and rotate to yield $|\vec{n}\rangle$

$\uparrow$ arbitrary direction

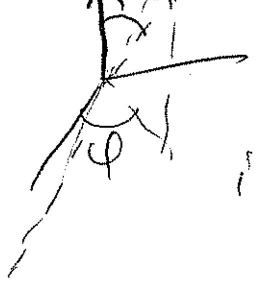
$\mathcal{D}(R)\,|\vec{z}\rangle = |\vec{n}\rangle \Rightarrow$

$$\cancel{|\vec{n}\rangle = \sum_{l,m} \mathcal{D}(R)\,|l,m\rangle\langle l,m|\vec{z}\rangle}$$

$$\underbrace{\langle l, m'|\vec{n}\rangle}_{= Y_l^{m'*}} = \sum_m \underbrace{\langle l,m'|\mathcal{D}(R)|l,m\rangle}_{\mathcal{D}^{(l)}_{m'm}} \underbrace{\langle l,m|\vec{z}\rangle}_{= \, ? \Leftrightarrow Y_l^m}$$

Recall: in a general case, $\mathcal{D}(R) = \mathcal{D}_z(\alpha)\mathcal{D}_y(\beta)\mathcal{D}_z(\gamma)$

What if we rotate a vector instead of a rigid body? $\Rightarrow$ need only one angle (in principle), but if we want to relate to $Y_l^m(\theta, \varphi) \rightarrow$ keep two angles $\Rightarrow \theta$ & φ $\Rightarrow$



$\leftarrow$ axis around which the rotation can be accomplished in one step instead of two

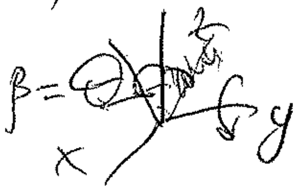
$$\alpha, \beta, \gamma \Longleftrightarrow \theta, \varphi$$

?

$\Rightarrow$

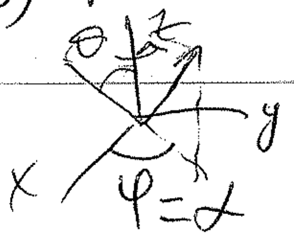
1) Rotate $|\vec{z}\rangle$ around z-axis by $\gamma \Rightarrow$ accomplishes nothing $\Rightarrow$ can set $\gamma = 0$

2) Rotate $|\vec{z}\rangle$ around y-axis by $\beta \Rightarrow$



$$\beta = \theta$$

3) Rotate around $z \Rightarrow \alpha = \varphi$



So, in this case $\mathcal{D}(R) = \mathcal{D}(\alpha = \varphi, \beta = \theta, \gamma = 0)$

Then, $$Y_\ell^{m'*}(\theta, \varphi) = \sum_m \mathcal{D}^{(\ell)}_{m'm}(\alpha = \varphi, \beta = \theta, \gamma = 0) \cdot \langle \ell, m | \vec{z}\rangle$$

What is $\langle \ell, m | \vec{z}\rangle$?

$$\Downarrow$$

$Y_\ell^{m*}(\theta = 0, \underset{\substack{\uparrow \\ \text{undetermined}}}{\varphi}) \Rightarrow$ vanish if $m \neq 0$!

So, $$\langle \ell, m | \vec{z}\rangle = \underbrace{Y_\ell^{0*}(\theta = 0)}_{\delta_{m0}} = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}\, P_\ell(\cos\theta)\Big|_{\theta=0} =$$

$$= \underline{\sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}}$$

Altogether:

$$Y_\ell^{m'*}(\theta,\varphi) = \sum_m \mathcal{D}^{(\ell)}_{m'm}(\alpha=\varphi,\beta=\theta,\gamma=0)\sqrt{\frac{2\ell+1}{4\pi}}\delta_{m0}$$

$$= \mathcal{D}^{(\ell)}_{m'0}(\alpha=\varphi,\beta=\theta,\gamma=0)\sqrt{\frac{2\ell+1}{4\pi}} \Rightarrow$$

$$\mathcal{D}^{(\ell)}_{m0}(\alpha,\beta,\gamma=0) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^{m*}(\theta,\varphi)\Big|_{\substack{\alpha=\varphi\\ \beta=\theta}}$$

Partial case: $m=0$ $\Rightarrow$

$$\mathcal{D}^{(\ell)}_{00}(\alpha,\beta,\gamma=0) = d^{(\ell)}_{00}(\underset{\beta}{\underset{\shortparallel}{\theta}}) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^{0*}(\theta) =$$

$$= P_\ell(\cos\theta)$$