QM II
Phys 652

Lecture # 19

Wigner-Eckart theorem

Selection rules; $\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = 0$ (since C.G. = 0)

(one of the consequences)

unless $m' = m + q$

$|j-k| \le j' \le j+k$ $\Rightarrow$ consequence of W.-E. theorem

Wigner-Eckart theorem

$$\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = \underbrace{\langle j, k ; m, q | j, k ; j', m' \rangle}_{\equiv \langle j, k; m, q | j', m' \rangle} \cdot \frac{\langle \alpha', j' \| T^{(k)} \| \alpha, j \rangle}{\sqrt{2j+1}}$$

Clebsch-Gordan coefficient

$\sqrt{2j+1}$ for convenience

$\langle \alpha', j' \| T^{(k)} \| \alpha, j \rangle$ — double bar or "reduced" matrix element $\Downarrow$ does not depend on m, m', q

Example Lecture #: selection rules for electric dipole allowed transitions

$$M_{fi} \sim \langle f | \vec{\epsilon} \cdot \vec{r} | i \rangle \sim \langle f | \overbrace{Y_1^q}^{T_q^{(1)}} | i \rangle \sim \langle n_f, l_f, m_f | Y_1^q | n_i, l_i, m_i \rangle$$

$\vec{\epsilon}$: polariz. of light

$\vec{r} \sim \sum \vec{\epsilon}_1^{\,-q} \, Y_1^q \, (-1)^q$

spherical basis

$\Downarrow$

$\boxed{m_f = m_i + q \quad l_f = l_i \pm 1}$

If we dealt with the quadrupole approximation ⇒ (2)

$$Q_{ij} = 3x_i x_j - r^2 \delta_{ij}$$ ← quadrupole tensor (traceless) (Cartesian)

$N=2$ ← rank

What are the selection rules for this quadrupole allowed transition? ⇒

Q_{ij} ⇒ present in terms of spherical tensors

Recall: arbitrary 2-nd rank $\underbrace{T_{(ij)}}_{\text{Cartesian}} = \underbrace{T^{(0)} + T^{(1)} + T^{(2)}}_{\text{spherical tensors}}$

So, for an arbitrary T ⇒

$\langle f | T | i \rangle \Rightarrow \Delta l = 0$ ← from scalar component $T^{(0)}$

$0; \pm 1$ ← from $T^{(1)}$

$0; \pm 1; \pm 2$ ← from $T^{(2)}$

Since Q_{ij} is a traceless symmetric tensor ⇒

$$Q_{ij} = T_{ij\,(s)} \Rightarrow T^{(2)} \Rightarrow \underline{\Delta l = \pm 2;\ \pm 1,\ 0}$$

(5 ind. components) Further selection ⇒ symmetry: $\Delta l =$ even ← parity!

<u>Example</u> $T^{(k)} = \vec{J}$, $k=1$ ⇒ find $\langle n', j' \| \vec{J} \| n, j \rangle$

Use W.-E. theorem!

$$\langle n', j', m' | J_q^{(1)} | n, j, m \rangle = \langle n', j' \| \vec{J} \| n, j \rangle \cdot$$

$$\cdot \langle j, 1; m, q | j', m' \rangle \cdot \frac{1}{\sqrt{2j+1}}$$

If $q=0 \Rightarrow \langle n',j',m'|\underbrace{J_0^{(1)}}_{J_z}|n,j,m\rangle =$

recall: for any vector $V_{q=0}^{(1)} = V_z$

$$= \langle n',j',m'|n,j,m\rangle \hbar m = \underline{\delta_{jj'}\delta_{n'n}\delta_{m'm}\hbar m}$$

Since $\langle n',j'\|\vec{J}\|n,j\rangle$ does <u>not</u> depend on q, m $\Rightarrow$ can consider the case of $q=0$ without a loss of generality $\Rightarrow$

$$\langle n',j'\|\vec{J}|n,j\rangle = \frac{\langle n',j',m'|J_{q=0}^{(1)}|n,j,m\rangle}{\langle j,1;m,0|j',m'\rangle}\sqrt{2j'+1} =$$

$$= \frac{\hbar m\,\delta_{jj'}\delta_{nn'}\delta_{mm'}}{\langle j,1;m,0|\underbrace{j'}_{j},\underbrace{m'}_{m}\rangle}\sqrt{2j+1} = \frac{\hbar m\sqrt{2j+1}\,\delta_{jj'}\delta_{nn'}}{\langle j,1;m,0|j,m\rangle} =$$

$$= \frac{\hbar j\sqrt{2j+1}}{\sqrt{j}}\sqrt{j+1}\,\delta_{jj'}\delta_{nn'} = \hbar\sqrt{j(j+1)}\sqrt{2j+1}\,\delta_{jj'}\delta_{nn'}$$

should be valid for any m $\Rightarrow$ take $m=j$, since we know $\langle j,1;j,0|j,j\rangle = \sqrt{\frac{j}{j+1}}$

Eq. (16.3)

Exercise question: What if we choose $m=0$?

Another consequence; the projection theorem (4)

(at $j=j'$)

For any vector operator $\vec{V}$ $\Rightarrow$

$$\langle \alpha', j, m' | V_q^{(1)} | \alpha, j, m \rangle = \frac{\langle \alpha', j, m | \vec{J} \cdot \vec{V} | \alpha, j, m \rangle}{\hbar^2 j(j+1)} \cdot$$

$$\cdot \langle j, m' | \underbrace{J_q^{(1)}}_{\Downarrow} | j, m \rangle$$

(Proof: see Sakurai pp. 241–242)

$$J_{\pm 1} = \mp \frac{J_x \pm i J_y}{\sqrt{2}} = \mp \frac{J_\pm}{\sqrt{2}}$$

$$J_0 = J_z$$

Another form of the projection theorem:

$$\frac{\langle n, j, m' | \vec{V} | n, j, m \rangle}{\langle n, j, m' | \vec{J} | n, j, m \rangle} = \frac{\langle n, j \| \vec{V} \| n, j \rangle}{\langle n, j \| \vec{J} \| n, j \rangle}$$

Example Find the mean value of the operator $\vec{\mu} = g_1 \vec{J}_1 + g_2 \vec{J}_2$ in the state characterized by the quantum numbers j_1, j_2, j, m if the total angular momentum is

$$\vec{J} = \vec{J}_1 + \vec{J}_2.$$

As before, $|j_1, j_2; j, m\rangle \equiv |j, m\rangle$

Need to find $\langle j, m | \vec{\mu} | j, m\rangle$

Consider $\vec{\mu} = (0, 0, \mu_z)$. $\vec{\mu}$ is a vector operator $\Rightarrow$

Use projection theorem $\Rightarrow$

$$\langle j, m | \underset{\substack{q=0 \\ k=1}}{\mu_z} | j, m\rangle = \frac{\langle j, m | \vec{J} \cdot \vec{\mu} | j, m\rangle}{\hbar^2 j(j+1)} \cdot \underbrace{\langle j, m | J_z | j, m\rangle}_{= \hbar m}$$

$$\langle j, m | \vec{J} \cdot \vec{\mu} | j, m\rangle = \langle j, m | g_1 \vec{J} \cdot \vec{J}_1 + g_2 \vec{J} \cdot \vec{J}_2 | j, m\rangle =$$

$$= \frac{1}{2} \langle j, m | g_1 (\vec{J}^2 + \vec{J}_1^2 - \vec{J}_2^2) + g_2 (\vec{J}^2 - \vec{J}_1^2 + \vec{J}_2^2) | j, m\rangle =$$

$$\vec{J} \cdot \vec{J}_1 = (\vec{J}_1 + \vec{J}_2) \cdot \vec{J}_1 =$$

$$= \vec{J}_1^2 + \frac{1}{2} (\vec{J}^2 - \vec{J}_1^2 - \vec{J}_2^2)$$

$$= \frac{\hbar^2}{2} \Big(g_1 \big(j(j+1) + j_1(j_1+1) - j_2(j_2+1) \big) + g_2 \big(j(j+1) -$$

$$- j_1(j_1+1) + j_2(j_2+1) \big) \Big) = \hbar^2 \left(\frac{g_1 + g_2}{2} j(j+1) + \frac{g_1 - g_2}{2} \cdot \right.$$

$$\left. \cdot \big(j_1(j_1+1) - j_2(j_2+1) \big) \right)$$

So, $\langle \mu_z \rangle = \frac{\hbar m}{\cancel{\hbar^2} j(j+1)} \cancel{\hbar^2} \left[\frac{g_1+g_2}{2} j(j+1) + \frac{g_1-g_2}{2} \cdot \left(j_1(j_1+1) - j_2(j_2+1)\right) \right]$

If $\vec{J}_1 = \vec{L}$, $\vec{J}_2 = \vec{S}$; $g_L = 1\mu_B$, $g_s = 2\mu_B \Rightarrow$ $\quad \vec{\mu} = (\vec{L} +$

$$\langle \mu_z \rangle = \cancel{\hbar} m \mu_B \left(\underbrace{\frac{3}{2} - \frac{1}{2} \frac{l(l+1) - s(s+1)}{j(j+1)}}_{=\, g_J \,\leftarrow\, \text{Lande g-factor}} \right)$$

Similarly,

$$\langle \mu_{x,y} \rangle = \frac{\langle \vec{J} \cdot \vec{\mu} \rangle}{\hbar^2 j(j+1)} \langle J_{x,y} \rangle = 0$$

$$J_x = \frac{J_+ + J_-}{2}$$

$$J_y = \frac{J_+ - J_-}{2i}$$

<u>Example of using g_J</u>

The electron is in $^2P_{3/2}$ state. In a weak magnetic field B what would be the energy splittings?

$$\Delta E = g_J \mu_B B m \; ; \quad {}^2P_{3/2} \Rightarrow l = 1, \; j = \frac{3}{2}, \; s = \frac{1}{2}$$

$$g_J = \frac{3}{2} - \frac{1}{2} \frac{1 \cdot 2 - \frac{1}{2} \cdot \frac{3}{2}}{\frac{3}{2} \cdot \frac{5}{2}} = \frac{4}{3} \Rightarrow |m| \le j \Rightarrow \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$$

$$\Delta E = \mu_B B \begin{cases} 2 \\ 2/3 \\ -2/3 \\ -2 \end{cases}$$