QM II
Phys 652

Lecture # 6

Angular momentum and rotations

What is angular momentum in QM? $\Rightarrow$
in most elementary books it's introduced as

$\vec{L} = \vec{r} \times \underline{\vec{p}}$ (operator equation)

Problem: spin $\vec{S}$ is also angular momentum, but it doesn't have anything to do with either $\vec{r}$ or $\vec{p}$!

$\Downarrow$

need a more general definition

$\Downarrow$

Angular momentum is the quantity that is conserved in systems with invariance under rotations

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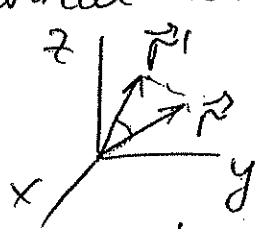
next 3 lectures will be devoted to developing understanding of this statement

General plan:

1) Consider geometrical rotations in 3D (coordinate) space

$\vec{r} \to \vec{r}\,'$



2) Generalise rotations to transformations of

State vectors in Hilbert space $\Rightarrow$

$$|\Psi\rangle \to |\Psi'\rangle, \text{ where } |\Psi'\rangle = \underbrace{\mathcal{D}}_{\text{rotation operator}}(R)|\Psi\rangle$$

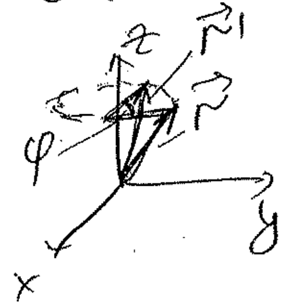
3) Show that $\mathcal{D}(R)$ is directly related to generalized angular momentum $\vec{J}$ (orbital angular momentum $\vec{L}$ and spin angular momentum $\vec{S}$ are partial cases of $\vec{J}$).

Let's start from geometrical rotations

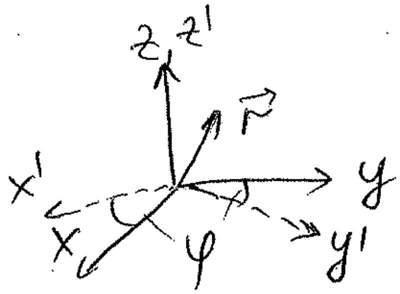
Consider a rotation about z-axis by angle φ.

$\Downarrow$

Two conventions are possible:



- active rotations: coordinate system (x, y, z) remains unchanged, rotation affects the physical system ($\vec{r} \to \vec{r}\,'$)



- passive rotations: physical system is fixed ($\vec{r}$), but the coordinate system is rotated ($x, y, z \to x', y', z'$)
  - * if rotation is around z, then z and z' coincide

We will consider active rotations

Recall classical mechanics:

$$\vec{r}\,' = R\,\vec{r}$$

R: 3x3 matrix in a 3D space

R is <u>orthogonal</u> (i.e. $RR^T = R^T R = I$), (R^T: transposed)

which ensures that

$$|\vec{r}\,'| = |\vec{r}|$$

For a rotation around z-axis by φ:

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (0 \le \varphi \le 2\pi)$$

Similarly, $$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

From a simple test with two boxes (or erasers) (see Sakurai p. 153), it's easy to show that

$$R_i(\varphi)\,R_j(\varphi') \neq R_j(\varphi')\,R_i(\varphi)$$

$(i \neq j)$ for arbitrary <u>finite</u> angles φ, φ'

i.e. finite angle rotations are <u>non-commutative</u>

Exercise: show this non-commutativity mathematically, i.e. using matrices $R_i(\varphi)$ $(i=x,y,z)$.

What about infinitesimal rotations, i.e. $\varphi \simeq \underset{\substack{\uparrow \\ \text{small parameter}}}{\varepsilon}$

Then $\cos\varphi \approx 1 - \frac{\varepsilon^2}{2}$; $\sin\varphi \sim \varepsilon$

Do we still have $R_i(\varepsilon) R_j(\varepsilon) \neq R_j(\varepsilon) R_i(\varepsilon)$? $\Rightarrow$

Consider $x=i$, $y=j$ $\Rightarrow$ $[R_i(\varepsilon), R_j(\varepsilon)] = [R_x(\varepsilon), R_y(\varepsilon)]$

$$\underset{\substack{\uparrow \\ \text{show!}}}{=} \underbrace{\begin{pmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$\Rightarrow$ so, if we ignore terms of ε^2 and higher (ε^n, $n>2$)

$\Downarrow$

infinitesimal rotations about different axes do commute

$\Uparrow$ can also be presented as $\underbrace{R_z(\varepsilon^2)} - \underset{\substack{\uparrow \\ \text{identity matrix}}}{1}$ (again, ignoring ε^n, $n>2$)

$$\begin{pmatrix} 1-\cancel{\frac{\varepsilon^4}{2}} & -\varepsilon^2 & 0 \\ \varepsilon^2 & 1-\cancel{\frac{\varepsilon^4}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Downarrow$ use this later!

What else do we know about rotations? $\Rightarrow$

- rotations by a finite angle around the same axis commute, i.e. $R_i(\varphi) R_i(\varphi') = R_i(\varphi') R_i(\varphi)$
- every finite rotation can be decomposed into an

infinite number of infinitesimal rotations

i.e. $R_i(\varphi+\varepsilon) = R_i(\varphi) R_i(\varepsilon) = R_i(\varepsilon) R_i(\varphi)$

Claim: The set of rotations R constitutes a group

To show that $\Rightarrow$ recall what a group is.

An abstract group is defined without a reference to any particular physical or mathematical system. The elements of the set $\{a, b, c, \dots\}$ form a group, if a combination $a \circ b$ of these elements, so that

called "multiplication", but does not mean "a times b"

it satisfies the following conditions:

1) The product $a \circ b$ is also an element of the group $G = \{a, b, c, \dots\}$ for all a's and b's (so-called closure condition)

2) The set $G = \{a, b, c, \dots\}$ contains a unit element e, which satisfies $a \circ e = e \circ a = a$ ← for arbitrary element of the group

3) For each element of the group, there exists an inverse element a^{-1}, so that $a^{-1} \circ a = a \circ a^{-1} = e$

4) The multiplication is associative, i.e.

$$(a \circ b) \circ c = a \circ (b \circ c)$$

The number of elements in a group, its order, can be finite, or denumerably or nondenumerably infinite.

Examples: finite groups: – symmetry groups of the regular solids

– permutation groups on a finite number of objects

infinite groups:

– denumerably: positive and negative integers (multiplication defined as addition)

– nondenumerably: set of real numbers (with respect to addition)

Consider a set of integers

$$N = \{0, \pm 1, \pm 2, \ldots\}$$ $\Rightarrow$ this set forms a denumerably infinite group $\Rightarrow$ let's prove it!

Need to ensure that the definitions 1) – 4) above are valid

1) Define a multiplication law $\Rightarrow$

try usual multiplication $\Rightarrow$ $a \circ b = ab$

Say, $a = 1$, $b = 2$ $\Rightarrow$ $1 \cdot 2 = 2$ $\Leftarrow$ integer $\downarrow$ belongs to the group $\Rightarrow$ seems OK $\Rightarrow$ go on

2) choose unit element e $\Rightarrow$ say, $e = 1$ $\Rightarrow$
$ae = ea = a$

$\uparrow$ integer $\Rightarrow$ OK! integer

go on

3) Inverse element a^{-1}, so that $a^{-1} \circ a = e$ $\Rightarrow$
$a^{-1} = \frac{1}{a}$ $\Rightarrow$ if $a = 2, 3$ $\Rightarrow$ $a^{-1} = \frac{1}{2}, \frac{1}{3}$, etc. $\quad e = 1$

not integers! don't belong to the group!! our multiplication law does not work!!

does it mean that our set N does not form a group? $\Rightarrow$ not quite $\Rightarrow$ look for another multiplication law $\Rightarrow$ back to square 1

1) Define multiplication law as an addition, i.e.
$a \circ b = a + b$

Say, $a = 1$, $b = 2$ $\Rightarrow$ $a \circ b = 1 + 2 = 3$ $\Leftarrow$ integer $\Rightarrow$ OK!

2) Unit element $e = 0 \Rightarrow a + 0 = 0 + a = a$

$\Downarrow$ for any integer $\downarrow$ OK!

3) Inverse element $\Rightarrow -a$

$a + (-a) = \underset{e}{0}$ for any a

$\Downarrow$ OK!

4) Associativity

$a + (b + c) = (a + b) + c$

valid for all a, b, c's

$\Downarrow$

Set N is a group with respect to addition!

Back to rotations

Operations of rotation form a group with respect to successive application of two rotations $\Rightarrow$

$$a \circ b = R_{\vec{n}}(\varphi) R_{\vec{n}'}(\varphi')$$

1) The product of two orthogonal matrices is another orthogonal matrix:

$$\left(R_{\vec{n}}(\varphi) R_{\vec{n}'}(\varphi') \right) \left(R_{\vec{n}}(\varphi) R_{\vec{n}'}(\varphi') \right)^T = R_{\vec{n}}(\varphi) \underbrace{R_{\vec{n}'}(\varphi')}_{}.$$

$$\underbrace{\cdot R_{\vec{n}'}^T(\varphi')}_{= I} R_{\vec{n}}^T(\varphi) = R_{\vec{n}}(\varphi) R_{\vec{n}}^T(\varphi) = I \quad \checkmark$$

2) Unit element $\Rightarrow$ the identity matrix $\Rightarrow$ rotation by $\varphi = 0$

$$R_{\vec{n}}(\varphi) R_{\vec{n}'}(0) = R_{\vec{n}}(\varphi) \quad \checkmark$$

3) inverse element $\Rightarrow$ rotation by $-\varphi \Rightarrow$ ✓

$$R_{\vec{n}}(\varphi)R_{\vec{n}}(-\varphi) = R_{\vec{n}}(0)$$

4) Associativity $R_{\vec{n}}(\varphi)\left(R_{\vec{n}'}(\varphi')R_{\vec{n}''}(\varphi'')\right) =$

$$= \left(R_{\vec{n}}(\varphi)R_{\vec{n}'}(\varphi')\right)R_{\vec{n}''}(\varphi'')$$

So, rotations form a group. What kind of a group?
The group of all 3×3 orthogonal matrices is denoted

~~as~~ $O(3)$

O ↑ orthogonal; (3) ↑ dimension of space

Since rotations preserve length <u>and</u> orientation of objects (e.g. vectors) $\Rightarrow$ from $\det R^T = \det R$

$$\Downarrow$$

$$(\det R)^2 = 1 \Rightarrow$$

choose $\underline{\det R = +1} \Leftarrow \det R = \pm 1$ inversion

$$\Downarrow$$

The sub-group of orthogonal matrices with $\det = +1$ is the <u>special</u> <u>orthogonal</u> group, denoted <u>$SO(3)$</u>.

$$\Downarrow$$

Important to know since we can derive properties of the system based on group theory!

An Abelian group is a group such that $a \circ b = b \circ a$ for all elements a & b.

Abelian groups are also called commutative groups.

Non-commutative groups are called non-Abelian.

Rotations in 3D form a non-Abelian group (see page 3)

HW:

1) Do translations in 3D form a group? What kind of a group?

2) What about rotations in 2D?

Reading assignment: pp. 152–155, 168–169 of Sakurai