MTH 252 — Lab 5
Methods of Integration

You may find the following trig identities useful:
\[
\sin(2\theta) = 2\sin\theta\cos\theta \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.
\]

1. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.
   (a) \[\int x \sin(x^2) \, dx\]
   (b) \[\int x^2 \sin x \, dx\]
   (c) \[\int \frac{x^2}{1 + x^2} \, dx\]
   (d) \[\int \frac{x}{(1 + x^2)^2} \, dx\]
   (e) \[\int x^3 e^{x^2} \, dx\]
   (f) \[\int \frac{\sin x}{2 + \cos x} \, dx\]

2. (a) Find \[\int \sin \theta \cos \theta \, d\theta\].
   (b) You probably solved part (a) by making the substitution \(w = \sin \theta\) or \(w = \cos \theta\). (If not, go back and do it that way.) Now find \[\int \sin \theta \cos \theta \, d\theta\] by making the other substitution.
   (c) There is yet another way of finding this integral which involves the trigonometric identities above. Find \[\int \sin \theta \cos \theta \, d\theta\] using one of these identities and then the substitution \(w = 2\theta\).
   (d) You should now have three different expressions for the indefinite integral \[\int \sin \theta \cos \theta \, d\theta\].
   Are they really different? Are they all correct? Explain.

3. (a) During the 1970s, ACME Widgets sold at a continuous rate of \(R = R_0 e^{0.15t}\) widgets per year, where \(t\) is time in years since January 1, 1970. Suppose they were selling widgets at a rate of 1000 per year on the first day of the decade. How many widgets did they sell during the decade? How many did they sell if the rate on January 1, 1970 was 150,000,000 widgets per year?
   (b) In the first case above (1000 widgets per year on January 1, 1970), how long did it take for half the widgets in the 1970s to be sold? In the second case (150,000,000 widgets per year on January 1, 1970), when had half the widgets in the 1970s been sold?
   (c) In 1980 ACME began an advertising campaign claiming that half the widgets it had sold in the previous ten years were still in use. Based on your answer to part (b), about how long must a widget last in order to justify this claim?

4. (a) Find the average value of the following functions over one cycle:
   (i) \(f(\theta) = \cos \theta\)
   (ii) \(g(\theta) = |\cos \theta|\)
   (iii) \(k(\theta) = (\cos \theta)^2\) (You may wish to use the trig identities above.)
   (b) Write the averages you have just found in ascending order. Using words and graphs, explain why the averages come out in the order they do.
5. The curves \( y = \sin x \) and \( y = \cos x \) cross each other infinitely often. What is the area of the region bounded by these two curves between two consecutive crossings?

6. Let \( E(x) = \int \frac{e^x}{e^x + e^{-x}} \, dx \) and \( F(x) = \int \frac{e^{-x}}{e^x + e^{-x}} \, dx \).
   (a) Calculate \( E(x) + F(x) \).
   (b) Calculate \( E(x) - F(x) \).
   (c) Use your results from parts (a) and (b) to calculate \( E(x) \) and \( F(x) \).

7. **Food for thought!**
   What, if anything, is wrong with the following calculation?
   \[
   \int_{-2}^{2} \frac{1}{x^2} \, dx = \left. -\frac{1}{x} \right|_{-2}^{2} = -\frac{1}{2} - \left( -\frac{1}{2} \right) = -1.
   \]