MTH 112
Extra Examples

1. (Problem 8, Rockswold) Since \( \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \) and \( \frac{\pi}{3} \) is in the interval \([0, \pi]\), \( \cos^{-1}\left(\frac{1}{2}\right) = ? \).

Solution. By definition \( \cos^{-1}\left(\frac{1}{2}\right) \) is an angle, let’s call it \( \theta \), in the interval \([0, \pi]\) such that \( \cos(\theta) = \frac{1}{2} \). As the problem has already explained, that angle is \( \theta = \frac{\pi}{3} \). That is, \( \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \).

FOLLOW UP QUESTION: Since \( \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \), why isn’t it that case that \( \cos^{-1}\left(\frac{1}{2}\right) = \frac{5\pi}{3} \)?

Solution. There are lots of possibilities for values of \( \theta \) that will satisfy \( \cos(\theta) = \frac{1}{2} \). We have decided that the range of the function \( f(x) = \cos^{-1}(x) \) will be \([0, \pi]\). Therefore \( \cos^{-1}(x) \) is always going to be a number (an angle) in the interval \([0, \pi]\). \( \frac{5\pi}{3} \) is not in this interval. Therefore \( \cos^{-1}\left(\frac{1}{2}\right) \neq \frac{5\pi}{3} \). Above we found that \( \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \).

2. (Problem 24, Rockswold) Evaluate \( \sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) \).

Solution. There are multiple ways that we can approach this problem. The first way is computationally. That is, first we compute \( \sin\left(\frac{\pi}{4}\right) \) and then we compute \( \sin^{-1} \) of our result.

\[
\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.
\]

The second way that we can approach this problem is conceptually. Let \( x \) represent \( \sin\left(\frac{\pi}{4}\right) \). That is, \( \sin\left(\frac{\pi}{4}\right) = x \). Thus \(-1 \leq x \leq 1\). We of course know that \( x = \frac{\sqrt{2}}{2} \). Now \( \sin^{-1}(x) \) is defined to be a number (an angle), say \( \theta \), in the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\) such that \( \sin(\theta) = x \). But we’ve already been shown that angle in this problem, it is \( \theta = \frac{\pi}{4} \). That is, \( \sin^{-1}(\sin\left(\frac{\pi}{4}\right)) = \frac{\pi}{4} \).

FOLLOW UP QUESTION. Why is it then not the case that \( \sin^{-1}(\sin\left(\frac{3\pi}{4}\right)) = \frac{3\pi}{2} \)?

Solution. \( \sin^{-1}(x) \) is defined to be the number \( y \) in the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\) such that \( \sin(y) = x \). \( \frac{3\pi}{2} \) is not in this interval, so \( \sin^{-1}(\sin\left(\frac{3\pi}{2}\right)) \) could not possibly be \( \frac{3\pi}{2} \). Instead we attack this problem computationally and see \( \sin\left(\frac{3\pi}{2}\right) = 1 \), so \( \sin^{-1}(\sin\left(\frac{3\pi}{2}\right)) = \sin^{-1}(1) \). Since \( \sin\left(\frac{\pi}{2}\right) = 1 \) and \( \frac{\pi}{2} \) is in the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\), we conclude that \( \sin^{-1}(1) = \frac{\pi}{2} \).
\[ \frac{\pi}{2}. \text{ That is, } \sin^{-1}(\sin(\frac{3\pi}{2})) = \frac{\pi}{2}. \]

**FOLLOW UP QUESTION 2.** What about evaluating \( \sin(\sin^{-1}(\frac{1}{2})) \)?

*Solution.* Again, we can approach this problem computationally or conceptually. Conceptually we see that \( \sin^{-1}(\frac{1}{2}) \) is defined to be the number \( y \) in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) such that \( \sin(y) = \frac{1}{2} \). Thus \( \sin(\sin^{-1}(\frac{1}{2})) \) must evaluate to be \( \frac{1}{2} \).

Computationally, we have \( \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6} \). Therefore \( \sin(\sin^{-1}(\frac{1}{2})) = \sin(\frac{\pi}{6}) = \frac{1}{2} \).

These two problems illustrate the property that we discussed in class:

\[ \sin^{-1}(\sin x) = x \text{ if (and only if) } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \]

and

\[ \sin(\sin^{-1} x) = x \text{ as long as } \sin^{-1}(x) \text{ is defined (as long as } -1 \leq x \leq 1). \]

**FOLLOW UP QUESTION 3.** In this problem we claimed that \( \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6} \). How did we calculate this since a calculator only gives a decimal answer?

*Solution.* The answer is a unit circle. I am looking for the angle \( \theta \) in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) such that \( \sin(\theta) = \frac{1}{2} \). We see that the (only) angle that makes this happen is \( \frac{\pi}{6} \). If I wanted to determine \( \sin^{-1}(x) \) and \( x \) does not appear as the sin value of any angle on my unit circle then I would probably only be able to get an approximate (decimal) answer from my calculator. The exception would be if the problem somehow gave me the information I needed to calculate it exactly.