NAME:____ **Exercise Set #1**

MATH 627, Fall 2004

1. Define the function $k(\cdot)$ by

$$k(x) = k_1,\ 0 < x < 1, \quad k(x) = k_2,\ 1 < x < 2\,.$$

Find the solution of the problem

$$-\partial(k(\cdot)\partial u(\cdot)) = 0 \text{ in } L^2(0,2), \quad u(0) = 1, u(2) = 0. \tag{1}$$

2. Let $V \equiv \{v \in L^2(0,2) : \partial v \in L^2(0,2) \text{ and } v(0) = 0\}$; the function $k(\cdot)$ is defined above. Let $F(\cdot) \in L^2(0,2)$ and $\lambda \in \mathbb{R}$ be given. Show that the function $u(\cdot)$ satisfies

$$u \in V \text{ and } \int_0^2 (k(x)\partial u(x)\partial v(x) + \lambda u(x)v(x))\,dx = \int_a^b F(x)v(x)\,dx\ , \qquad v \in V\ , \tag{2}$$

if and only if it satisfies the *interface problem*

$$\begin{gathered} u(0) = 0, \quad -\partial(k_1\partial u(x)) + \lambda u(x) = F(x), \quad 0 < x < 1, \\ u(1^-) = u(1^+), \quad k_1\partial u(1^-) = k_2\partial u(1^+), \\ -\partial(k_2\partial u(x)) + \lambda u(x) = F(x), \quad 1 < x < 2, \quad k_2\partial u(2) = 0\,. \end{gathered}$$

3. Nonhomogeneous Dirichlet conditions.

(a) Show the solution to

$$u \in H^1(0,\ell) : u(0) = f_1\ , \quad u(\ell) = f_2\ , \quad -\partial^2 u = F$$

satisfies (2.9) where

$$a(u,v) = \int_0^\ell \partial u \partial v\,dx\ , \qquad f(v) = \int_0^\ell Fv\,dx \qquad u,v \in H^1(0,\ell)\ ,$$

and

$$K = \{v \in H^1(0,\ell) : v(0) = f_1, v(\ell) = f_2\}\ .$$

(b) Show that the set K is the translate of the subspace $H_0^1(0,\ell)$ by the function

$$u_0(x) = (\ell - x)f_1/\ell + xf_2/\ell\ ,$$

and this variational inequality is equivalent to

$$u \in K : a(u,\varphi) = f(\varphi)\ , \qquad \varphi \in H_0^1(0,\ell)\ .$$

(c) Show this problem is actually a “linear” problem for the unkown $w \equiv u - u_0$ in the form

$$w \in H_0^1(0,\ell) : a(w,\varphi) = f(\varphi) - a(u_0,\varphi)\ , \qquad \varphi \in H_0^1(0,\ell)\ ,$$

and thus it is well-posed by the Lax-Milgram Theorem 2.2.