1. Abstract Cauchy Problem

Suppose that V is Hilbert space and that $\mathcal{A} : V \to V'$ is a linear monotone operator, that is,

$$\mathcal{A}u(u) \geq 0 \text{ for all } u \in V.$$

Let $\mathcal{B} : V \to V'$ be continuous, linear, symmetric and strictly positive. Then $\mathcal{B}(\cdot)(\cdot)$ is a (continuous) scalar product on V, and we denote the space V with the corresponding norm $(\mathcal{B}(\cdot)(\cdot))^{1/2}$ by W_b. Then the imbedding $V \hookrightarrow W_b$ is continuous, we have $W_b' \subset V'$, and the injection is continuous. The dual W_b' is a Hilbert space, and we have

$$f(u) = (f, \mathcal{B}u)_{W_b'}, \quad f \in W_b', u \in V\,.$$

Define the operator $\mathbb{A} : \mathrm{Dom}(\mathbb{A}) \to W_b'$ with domain $\mathrm{Dom}(\mathbb{A}) \subset W_b'$ by

$$\mathbb{A}(v) = f \iff \text{ for some } u \in V,\ \mathcal{B}u = v \text{ and } f = \mathcal{A}(u).$$

Then for any such pair, $[v, f] \in \mathbb{A}$ we have

$$(f, v)_{W_b'} = f(u)\,,$$

and since $\mathcal{A}$ is monotone, it follows that $\mathbb{A}$ is W_b'-accretive.

Remark 1.1. *If we replace W_b by its* completion, *all the above holds for the continuous extension of $\mathcal{B}$, and moreover $\mathcal{B} : W_b \to W_b'$ is the Riesz map. We also see that $\mathbb{A}$ is just the composition $\mathcal{A} \circ \mathcal{B}^{-1}$ with range restricted to W_b'.*

The equation $v + \mathbb{A}(v) = f$ in W_b' is equivalent to

$$u \in V : \ \mathcal{B}u = v,\ f - v = \mathcal{A}(u),$$

that is, $u \in V : \ \mathcal{A}(u) + \ B(u) = f \in W_b'$, so $\mathrm{Rg}(I + \mathbb{A}) = \mathrm{Rg}(\mathcal{B} + \mathcal{A}) \cap W_b' \subset V'$. This shows that

Lemma 1.1. *$\mathbb{A}$ is m-accretive on W_b' if $\mathrm{Rg}(\mathcal{B} + \mathcal{A}) \supset W_b'$.*

From the semigroup generation theorem, we obtain the following.

Theorem 1.2. *If $u_0 \in V$ with $\mathcal{A}(u_0) \in W_b'$, then there exists a unique function $u : [0, \infty) \to V$ with $\mathcal{B}u \in C^1([0, \infty); W_b')$ and*

$$\frac{d}{dt}\mathcal{B}u(t) + \mathcal{A}(u(t)) = 0 \text{ for all } t \geq 0, \tag{1a}$$

$$\mathcal{B}u(0) = \mathcal{B}u_0 \text{ in } W_b'. \tag{1b}$$

Exercise 1. *Let $V = \{w \in H^1(0, \ell) : \ w(0) = 0\}$, and set*

$$\mathcal{A}u(v) = \int_0^\ell \partial u(x)\, \partial v(x)\, dx\,, \qquad u,\ v \in V$$

Define the operator $\mathcal{B}$ by

$$\mathcal{B}u(v) = \int_0^\ell \rho(x) u(x)\, v(x)\, dx\,, \qquad u,\ v \in V\,,$$

where the function $\rho(\cdot) \in L^\infty(0, \ell)$ satisfies $\rho(x) > 0$ a.e. in $(0, \ell)$.

Show that $\mathcal{B}$ is continuous on V. Characterize each of W_b and W_b'. State which initial-boundary-value problem has been solved by Theorem 1.2, and verify your claim.

Exercise 2. *Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by*

$$\mathcal{B}u(v) = \int_0^\ell \rho(x)u(x)\,v(x)\,dx + \rho_0 u(\ell)\,v(\ell)\,, \qquad u,\ v \in V\,,$$

where $\rho_0 > 0$ is given.

Exercise 3. *Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by*

$$\mathcal{B}u(v) = \int_0^\ell (u(x)\,v(x) + k\,\partial u(x)\,\partial v(x))\,dx\,, \qquad u,\ v \in V\,,$$

where $k > 0$ is given.