The Diffusion Equation

We shall develop a representation of the solution of the *initial-value problem*

$$u_t(x,t) = u_{xx}(x,t), \quad -\infty < x < +\infty, \quad t > 0, \tag{1a}$$

$$u(x,0) = u_0(x). \tag{1b}$$

Note that if $U(x,t)$ is a solution of (1a), then so also is the translate $U(x-y,t)$ for any fixed $y \in \mathbb{R}$ as well as the linear combinations $\sum_j U(x-y_j,t)u_0(y_j)$ or even the integral $\int U(x-y,t)u_0(y)\,dy$. These remarks suggest that we should try first to find a solution $K(x,t)$ of the initial-value problem with the initial value $K(\cdot,0) = \delta$. (Then the last integral formally gives the initial condition (1b).) To do so, we shall solve the problem for a solution $U(x,t)$ with the initial condition $U(x,0) = H(x)$, where $H(\cdot)$ is the Heaviside *function*. Then we can compute $K(x,t) = \frac{\partial}{\partial x}U(x,t)$, since the derivative of a solution of (1a) is also a solution.

Note that if $u(x,t)$ is a solution of the initial-value problem (1) with $u_0(x) = H(x)$, then so is $u(ax, a^2t)$ for any number $a > 0$, since $H(\cdot)$ is invariant under dilation. The *uniqueness* of a solution suggests that $u(\cdot,\cdot)$ has the form

$$U(x,t) = g(s), \quad s = \frac{x}{\sqrt{t}}, \quad t > 0\,.$$

Substituting this into (1a), we obtain the equivalent form

$$g''(s) + \frac{1}{2}s\,g'(s) = 0,$$

so we find that

$$g(s) = c_1 \int e^{-\frac{s^2}{4}}\,ds + c_2\,,$$

and the corresponding solutions of the equation (1a) are given by

$$U(x,t) = c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4}}\,ds + c_2\,.$$

In order to get the initial condition $U(x,0) = H(x)$, we need

$$x > 0: \quad 1 = c_1\sqrt{\pi} + c_2,$$
$$x < 0: \quad 0 = -c_1\sqrt{\pi} + c_2,$$

so we have $c_1 = \frac{1}{2\sqrt{\pi}}$ and $c_2 = \frac{1}{2}$. That is, we obtain the solution of the problem in the form

$$U(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4}}\,ds + \frac{1}{2}\,.$$

We then obtain the desired solution with initial condition δ from the derivative

$$K(x,t) = U_x(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}\,.$$

Finally, we calculate

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, u_0(y)\, dy$$
$$= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y} U(x-y,t)\, u_0(y)\, dy = \int_{-\infty}^{\infty} U(x-y,t)\, u_0'(y)\, dy\,,$$

and so we obtain the initial value

$$u(x,0) = \int_{-\infty}^{\infty} H(x-y)\, u_0'(y)\, dy = \int_{-\infty}^{x} u_0'(y)\, dy = u_0(x).$$

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