1. Abstract Cauchy Problem

Suppose that V is Hilbert space and that $\mathcal{A} : V \to V'$ is a linear monotone operator, that is,

$$\mathcal{A}u(u) \geq 0 \text{ for all } u \in V.$$

Let $\mathcal{B} : V \to V'$ be continuous, linear, symmetric and strictly positive. Then $\mathcal{B}(\cdot)(\cdot)$ is a (continuous) scalar product on V, and we denote the space V with the corresponding norm $(\mathcal{B}(\cdot)(\cdot))^{1/2}$ by W_b. Then the imbedding $V \hookrightarrow W_b$ is continuous, we have $W_b' \subset V'$, and the injection is continuous. The dual W_b' is a Hilbert space, and we have

$$f(u) = (f, \mathcal{B}u)_{W_b'}, \quad f \in W_b', u \in V\,.$$

Define the operator $\mathbb{A} : \mathrm{Dom}(\mathbb{A}) \to W_b'$ with domain $\mathrm{Dom}(\mathbb{A}) \subset W_b'$ by

$$\mathbb{A}(v) = f \iff \text{ for some } u \in V,\ \mathcal{B}u = v \ \text{ and } \ f = \mathcal{A}(u).$$

Then for any such pair, $[v, f] \in \mathbb{A}$ we have

$$(f, v)_{W_b'} = f(u)\,,$$

and since $\mathcal{A}$ is monotone, it follows that $\mathbb{A}$ is W_b'-accretive.

Remark 1.1. *If we replace W_b by its* completion, *all the above holds for the continuous extension of $\mathcal{B}$, and moreover $\mathcal{B} : W_b \to W_b'$ is the Riesz map. We also see that $\mathbb{A}$ is just the composition $\mathcal{A} \circ \mathcal{B}^{-1}$ with range restricted to W_b'.*

The equation $v + \mathbb{A}(v) = f$ in W_b' is equivalent to

$$u \in V :\ \mathcal{B}u = v,\ f - v = \mathcal{A}(u),$$

that is, $u \in V :\ \mathcal{A}(u) +\ B(u) = f \in W_b'$, so $\mathrm{Rg}(I + \mathbb{A}) = \mathrm{Rg}(\mathcal{B} + \mathcal{A}) \cap W_b' \subset V'$. This shows that

Lemma 1.1. *$\mathbb{A}$ is m-accretive on W_b' if $\mathrm{Rg}(\mathcal{B} + \mathcal{A}) \supset W_b'$.*

From the semigroup generation theorem, we obtain the following.

Theorem 1.2. *If $u_0 \in V$ with $\mathcal{A}(u_0) \in W_b'$, then there exists a unique function $u : [0, \infty) \to V$ with $\mathcal{B}u \in C^1([0, \infty); W_b')$ and*

$$\frac{d}{dt}\mathcal{B}u(t) + \mathcal{A}(u(t)) = 0 \textit{ for all } t \geq 0, \tag{1a}$$

$$\mathcal{B}u(0) = \mathcal{B}u_0 \textit{ in } W_b'. \tag{1b}$$

Exercise 1. *Let $V = \{w \in H^1(0, \ell) :\ w(0) = 0\}$, and set*

$$\mathcal{A}u(v) = \int_0^\ell \partial u(x)\, \partial v(x)\, dx\,, \qquad u,\ v \in V$$

Define the operator $\mathcal{B}$ by

$$\mathcal{B}u(v) = \int_0^\ell \rho(x) u(x)\, v(x)\, dx\,, \qquad u,\ v \in V\,,$$

where the function $\rho(\cdot) \in L^\infty(0, \ell)$ satisfies $\rho(x) > 0$ a.e. in $(0, \ell)$.

Show that $\mathcal{B}$ is continuous on V. Characterize each of W_b and W_b'. State which initial-boundary-value problem has been solved by Theorem 1.2, and verify your claim.

Exercise 2. *Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by*

$$\mathcal{B}u(v) = \int_0^\ell \rho(x)u(x)\,v(x)\,dx + \rho_0 u(\ell)\,v(\ell)\,, \qquad u,\ v \in V\,,$$

where $\rho_0 > 0$ is given.

Exercise 3. *Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by*

$$\mathcal{B}u(v) = \int_0^\ell (u(x)\,v(x) + k\,\partial u(x)\,\partial v(x))\,dx\,, \qquad u,\ v \in V\,,$$

where $k > 0$ is given.

1.1. **The Complementary Form.** Define $\mathbb{A}$ by

$$\mathbb{A}(v) = f \iff \text{for some } u \in V,\ \mathcal{B}u = f \text{ and } v = \mathcal{A}(u).$$

Then for any such pair, $[v, f] \in \mathbb{A}$ we have

$$(f, v)_{W_b'} = v(u) = \mathcal{A}u(u)\,,$$

so if $\mathcal{A}$ is monotone, then $\mathbb{A}$ is W_b'-accretive.

The equation $v + \mathbb{A}(v) = f$ in W_b' is equivalent to

$$u \in V:\ v + \mathcal{B}u = f,\ v = \mathcal{A}(u),$$

so $\mathrm{Rg}(I + \mathbb{A}) = \mathrm{Rg}(\mathcal{B} + \mathcal{A}) \cap W_b' \subset V'$. This shows that

Lemma 1.3. *$\mathbb{A}$ is m-accretive on W_b' if $\mathrm{Rg}(\mathcal{B} + \mathcal{A}) \supset W_b'$.*

Theorem 1.4. *If $u_0 \in V$ and $v_0 = \mathcal{A}(u_0) \cap W_b'$, then there exists a unique function $u : [0, \infty) \to V$ with $\mathcal{B}u \in C([0,\infty); W_b')$ and $\mathcal{A}(u(\cdot)) \in C^1([0,\infty); W_b')$ for which*

$$\frac{d}{dt}\mathcal{A}(u(t)) + \mathcal{B}(u(t)) = 0 \text{ for all } t \geq 0, \tag{2a}$$

$$\mathcal{A}(u(0)) = v_0 \text{ in } W_b'. \tag{2b}$$

Remark 1.2. *Note that the roles of $\mathcal{A}$ and $\mathcal{B}$ have been reversed. The possibly unsymmetric operator now appears under the time derivative.*

1.2. **The Strong Form.** Let $u : [0, \infty) \to V$ with $\mathcal{B}u \in C([0,\infty); W_b')$ and $\mathcal{A}(u(\cdot)) \in C^1([0,\infty); W_b')$ be a pair as above for which

$$\frac{d}{dt}\mathcal{A}(u(t)) + \mathcal{B}(u(t)) = 0, \text{ for all } t \geq 0,$$
$$\mathcal{A}(u(0)) = v_0 \text{ in } W_b'.$$

Assume that $\mathcal{B} : W_b \to W_b'$ is *surjective.* (This can be accomplished by completing the scalar product space W_b and extending $\mathcal{B}$ by continuity.) Choose $w(t) \in W_b :\ \mathcal{B}(w(t)) =$

$\int_0^t \mathcal{B}(u(s))\,ds - v_0$. Then we have $w \in C^1([0,\infty); W_b)$

$$\frac{d}{dt}\mathcal{B}(w(t)) = \mathcal{B}(u(t))$$
$$\mathcal{A}(u(t)) + \mathcal{B}(w(t)) = 0 \text{ for all } t \geq 0,$$
$$\mathcal{A}(u(0)) = v_0.$$

In particular, since $\mathcal{B}$ is *injective*, we have $u(t) = w'(t) \in V$ and

$$\mathcal{A}(w'(t)) + \mathcal{B}(w(t)) = 0\,.$$

2. The Wave Equation

We want to resolve an appropriate initial-value problem for the *wave equation*

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = 0\,,$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are given operators in $\mathcal{L}(V, V')$. As above we write this as a system

$$\mathcal{A}u' \qquad - \mathcal{A}v = 0,$$
$$\mathcal{C}v' + \mathcal{A}u + \mathcal{B}v = 0.$$

Thus we see that the wave equation can be written in the form (1a) as

$$\frac{d}{dt}\begin{pmatrix}\mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix} + \begin{pmatrix}0 & -\mathcal{A}\\ \mathcal{A} & \mathcal{B}\end{pmatrix}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

The preceding suggests the following approach. Suppose that V is Hilbert space and that $\mathcal{B} : V \to V'$ is a linear monotone operator, that is,

$$\mathcal{B}u(u) \geq 0 \text{ for all } u \in V.$$

Let $\mathcal{A}$, $\mathcal{C} \in \mathcal{L}(V, V')$ both be continuous, linear, symmetric and strictly positive, so they determine as before a pair of scalar products on V, and we denote the completions of the space V with the corresponding norms by W_a and W_c, respectively. Then the imbeddings $V \hookrightarrow W_a$ and $V \hookrightarrow W_c$ are continuous, and we have $W_a' \subset V'$, $W_c' \subset V'$ with continuous injections. We define the matrix operators

$$\mathbb{B} = \begin{pmatrix}\mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix}0 & -\mathcal{A}\\ \mathcal{A} & \mathcal{B}\end{pmatrix}$$

on the product space $\mathbb{V} = V \times V$ into its dual $\mathbb{V}' = V' \times V'$. Then the continuous, linear, symmetric and strictly positive operator $\mathbb{B}$ is a scalar product on $\mathbb{V}$ for which the completion is the product space $W_a \times W_c$, and the above is in the form of (1a), and so Theorem 1.2 applies.

Let's check hypotheses. First, $\mathbb{A} : \mathbb{V} \to \mathbb{V}'$ is monotone, since $\mathcal{B} : V \to V'$ is monotone. Next, the range condition is satisfied if we can always solve

$$\lambda\mathbb{B}\mathbf{u} + \mathbb{A}\mathbf{u} = \mathbf{f} = [f_a, f_c] \in W_a' \times W_c'$$

for $\mathbf{u} = [u, v]$, that is,

$$\lambda\begin{pmatrix}\mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}0 & -\mathcal{A}\\ \mathcal{A} & \mathcal{B}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}f_a\\ f_c\end{pmatrix}.$$

This system is equivalent to the single equation

$$\lambda^2 \mathcal{C}\mathbf{v} + \lambda \mathcal{B}\mathbf{v} + \mathcal{A}\mathbf{v} = \lambda f_c - f_a \,,$$

and a sufficient condition for this is that $\mathrm{Rg}(\lambda^2 \mathcal{C} + \lambda \mathcal{B} + \mathcal{A}) \supset V'$.

Remark 2.1. *A sufficient condition for this range condition is that $\mathcal{A}$ be V-elliptic, and in that case we have $W_a = V$.*

The first component, $u(\cdot)$, satisfies $u \in C^1([0,\infty), W_a) \cap C^2([0,\infty), W_c)$ and

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = 0 \text{ in } W_c' \,. \tag{3}$$

The second component, $v(\cdot)$, satisfies $v \in C^1([0,\infty), W_c)$ with $\mathcal{C}v'(\cdot) + \mathcal{B}v(\cdot) \in C^1([0,\infty), W_a')$ and

$$(\mathcal{C}v'(t) + \mathcal{B}v(t))' + \mathcal{A}v(t) = 0 \text{ in } W_a' \,. \tag{4}$$

Note that we usually have $\mathcal{W}_a \subset \mathcal{W}_c$, and then (3) is *stronger* than (4).