1. Abstract Cauchy Problem

Suppose that $V$ is Hilbert space and that $\mathcal{A} : V \to V'$ is a linear monotone operator, that is,
$$\mathcal{A}u(u) \geq 0 \text{ for all } u \in V.$$Let $\mathcal{B} : V \to V'$ be continuous, linear, symmetric and strictly positive. Then $\mathcal{B}(\cdot)(\cdot)$ is a (continuous) scalar product on $V$, and we denote the space $V$ with the corresponding norm $(\mathcal{B}(\cdot)(\cdot))^{1/2}$ by $W$. Then the imbedding $V \hookrightarrow W$ is continuous, we have $W \subset V'$, and the injection is continuous. The dual $W^*$ is a Hilbert space, and we have
$$f(u) = (f, Bu)_{W^*}, \quad f \in W, u \in V.$$Define the operator $\mathcal{A} : \text{Dom}(\mathcal{A}) \to W'$ with domain $\text{Dom}(\mathcal{A}) \subset W'$ by
$$\mathcal{A}(v) = f \iff \text{for some } u \in V, \, Bu = v \text{ and } f = \mathcal{A}(u).$$Then for any such pair, $[v, f] \in \mathcal{A}$ we have
$$(f, v)_{W'} = f(u),$$and since $\mathcal{A}$ is monotone, it follows that $\mathcal{A}$ is $W^*$-accretive.

**Remark 1.1.** If we replace $W$ by its completion, all the above holds for the continuous extension of $\mathcal{B}$, and moreover $\mathcal{B} : W \to W'$ is the Riesz map. We also see that $\mathcal{A}$ is just the composition $\mathcal{A} \circ \mathcal{B}^{-1}$ with range restricted to $W'$.

The equation $v + \mathcal{A}(v) = f$ in $W$ is equivalent to
$$u \in V : \quad Bu = v, \quad f - v = \mathcal{A}(u),$$that is, $u \in V : \quad \mathcal{A}(u) + B(u) = f \in W'$, so $\text{Rg}(I + \mathcal{A}) = \text{Rg}(\mathcal{B} + \mathcal{A}) \cap W'. This shows that

**Lemma 1.1.** $\mathcal{A}$ is $m$-accretive on $W'$ if $\text{Rg}(\mathcal{B} + \mathcal{A}) \subset W'$.

From the semigroup generation theorem, we obtain the following.

**Theorem 1.2.** If $u_0 \in V$ with $\mathcal{A}(u_0) \in W'$, then there exists a unique function $u : [0, \infty) \to V$ with $Bu \in C^1([0, \infty); W)$ and

| (1a) | $\frac{d}{dt}Bu(t) + \mathcal{A}(u(t)) = 0$ for all $t \geq 0,$ |
| (1b) | $Bu(0) = Bu_0$ in $W'$. |

**Exercise 1.** Let $V = \{w \in H^1(0, \ell) : \ w(0) = 0\}$, and set
$$\mathcal{A}u(v) = \int_0^\ell \partial u(x) \partial v(x) \, dx, \quad u, \ v \in V$$Define the operator $\mathcal{B}$ by
$$\mathcal{B}u(v) = \int_0^\ell \rho(x)u(x) v(x) \, dx, \quad u, \ v \in V,$$where the function $\rho(\cdot) \in L^\infty(0, \ell)$ satisfies $\rho(x) > 0$ a.e. in $(0, \ell).$
Show that $\mathcal{B}$ is continuous on $V$. Characterize each of $W_b$ and $W_b'$. State which initial-boundary-value problem has been solved by Theorem 1.2, and verify your claim.

**Exercise 2.** Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by

$$\mathcal{B} u(v) = \int_0^\ell \rho(x) u(x) v(x) \, dx + \rho(u(\ell)) v(\ell), \quad u, \, v \in V,$$

where $\rho_0 > 0$ is given.

**Exercise 3.** Repeat Exercise 1 with the operator $\mathcal{B}$ replaced by

$$\mathcal{B} u(v) = \int_0^\ell (u(x) v(x) + k \partial u(x) \partial v(x)) \, dx, \quad u, \, v \in V,$$

where $k > 0$ is given.

1.1. The Complementary Form. Define $\mathcal{A}$ by

$$\mathcal{A}(v) = f \iff \text{for some } u \in V, \, \mathcal{B} u = f \text{ and } v = \mathcal{A}(u).$$

Then for any such pair, $[v, f] \in \mathcal{A}$ we have

$$(f, v)_{W_b'} = v(u) = \mathcal{A} u(u),$$

so if $\mathcal{A}$ is monotone, then $\mathcal{A}$ is $W_b'$-accractive.

The equation $v + \mathcal{A}(v) = f$ in $W_b'$ is equivalent to

$$u \in V: \, v + \mathcal{B} u = f, \, v = \mathcal{A}(u),$$

so $\text{Rg}(I + \mathcal{A}) = \text{Rg}(\mathcal{B} + \mathcal{A}) \cap W_b' \subset V'$. This shows that

**Lemma 1.3.** $\mathcal{A}$ is $m$-accractive on $W_b'$ if $\text{Rg}(\mathcal{B} + \mathcal{A}) \subset W_b'$.

**Theorem 1.4.** If $u_0 \in V$ and $v_0 = \mathcal{A}(u_0) \cap W_b'$, then there exists a unique function $u : [0, \infty) \to V$ with $\mathcal{B} u \in C([0, \infty); W_b')$ and $\mathcal{A}(u(\cdot)) \in C^1([0, \infty); W_b')$ for which

(2a) \quad \frac{d}{dt} \mathcal{A}(u(t)) + \mathcal{B}(u(t)) = 0 \text{ for all } t \geq 0,

(2b) \quad \mathcal{A}(u(0)) = v_0 \text{ in } W_b'.

**Remark 1.2.** Note that the roles of $\mathcal{A}$ and $\mathcal{B}$ have been reversed. The possibly unsymmetric operator now appears under the time derivative.

1.2. The Strong Form. Let $u : [0, \infty) \to V$ with $\mathcal{B} u \in C([0, \infty); W_b')$ and $\mathcal{A}(u(\cdot)) \in C^1([0, \infty); W_b')$ be a pair as above for which

$$\frac{d}{dt} \mathcal{A}(u(t)) + \mathcal{B}(u(t)) = 0, \text{ for all } t \geq 0,$$

$$\mathcal{A}(u(0)) = v_0 \text{ in } W_b'.$$

Assume that $\mathcal{B} : W_b \to W_b'$ is surjective. (This can be accomplished by completing the scalar product space $W_b$ and extending $\mathcal{B}$ by continuity.) Choose $w(t) \in W_b : \, \mathcal{B}(w(t)) =$
\[ \int_0^t B(u(s)) \, ds - v_0. \] Then we have \( w \in C^1([0, \infty); W_b) \)

\[ \frac{d}{dt} B(w(t)) = B(u(t)) \]

\[ \mathcal{A}(u(t)) + B(w(t)) = 0 \text{ for all } t \geq 0, \]

\[ \mathcal{A}(u(0)) = v_0. \]

In particular, since \( B \) is injective, we have \( u(t) = w'(t) \in V \) and

\[ \mathcal{A}(w'(t)) + B(w(t)) = 0. \]

2. The Wave Equation

We want to resolve an appropriate initial-value problem for the wave equation

\[ Cu''(t) + Bu'(t) + Au(t) = 0, \]

where \( A, B, C \) are given operators in \( \mathcal{L}(V, V') \). As above we write this as a system

\[ A u' - A v = 0, \]

\[ C v' + A u + B v = 0. \]

Thus we see that the wave equation can be written in the form (1a) as

\[ \frac{d}{dt} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 & -A \\ A & B \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

The preceding suggests the following approach. Suppose that \( V \) is Hilbert space and that \( B : V \to V' \) is a linear monotone operator, that is,

\[ Bu(u) \geq 0 \text{ for all } u \in V. \]

Let \( A, C \in \mathcal{L}(V, V') \) both be continuous, linear, symmetric and strictly positive, so they determine as before a pair of scalar products on \( V \), and we denote the completions of the space \( V \) with the corresponding norms by \( W_a \) and \( W_c \), respectively. Then the imbeddings \( V \hookrightarrow W_a \) and \( V \hookrightarrow W_c \) are continuous, and we have \( W_a' \subset V', W_c' \subset V' \) with continuous injections. We define the matrix operators

\[ \mathbb{B} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 0 & -A \\ A & B \end{pmatrix} \]

on the product space \( \mathbb{V} = V \times V \) into its dual \( \mathbb{V}' = V' \times V' \). Then the continuous, linear, symmetric and strictly positive operator \( \mathbb{B} \) is a scalar product on \( \mathbb{V} \) for which the completion is the product space \( W_a \times W_c \), and the above is in the form of (1a), and so Theorem 1.2 applies.

Let’s check hypotheses. First, \( \mathbb{A} : \mathbb{V} \to \mathbb{V}' \) is monotone, since \( B : V \to V' \) is monotone. Next, the range condition is satisfied if we can always solve

\[ \lambda \mathbb{B} u + \mathbb{A} u = f = [f_a, f_c] \in W_a' \times W_c' \]

for \( u = [u, v] \), that is,

\[ \lambda \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -A \\ A & B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_a \\ f_c \end{pmatrix}. \]
This system is equivalent to the single equation
\[ \lambda^2 C \nu + \lambda B \nu + \mathcal{A} \nu = \lambda f_a - f_a, \]
and a sufficient condition for this is that \( \text{Rg}(\lambda^2 C + \lambda B + \mathcal{A}) \supset V'. \)

**Remark 2.1.** A sufficient condition for this range condition is that \( \mathcal{A} \) be \( V \)-elliptic, and in that case we have \( W_a = V. \)

The first component, \( u(\cdot) \), satisfies \( u \in C^1([0, \infty), W_a) \cap C^2([0, \infty), W_c) \) and
\[ Cu''(t) + Bu'(t) + \mathcal{A}u(t) = 0 \text{ in } W_c'. \]
The second component, \( v(\cdot) \), satisfies \( v \in C^1([0, \infty), W_c) \) with \( Cu'(\cdot) + Bu(\cdot) \in C^1([0, \infty), W_a' \) and
\[ (Cu'(t) + Bu(t))' + \mathcal{A}v(t) = 0 \text{ in } W_a'. \]
Note that we usually have \( W_a \subset W_c \), and then (3) is stronger than (4).