Aliasing in Fourier Analysis

Optional Assessment; Practically Important

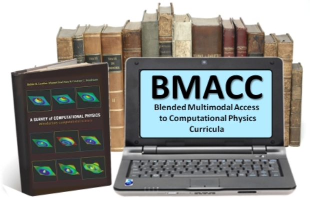
Rubin H Landau

Sally Haerer, Producer-Director

Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

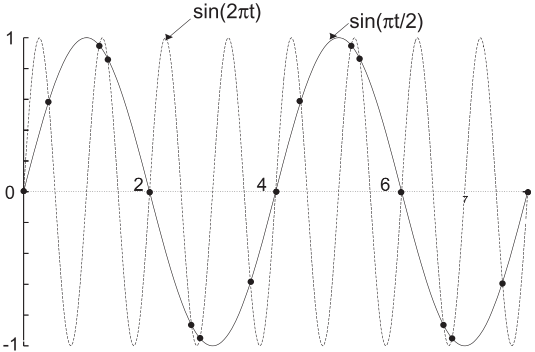
with Support from the National Science Foundation

Course: **Computational Physics II**



Outline

What is Aliasing?



- Signal contains 2 functions
- $\sin(\pi t/2)$ & $\sin(2\pi t)$
- Distinguish?
- Interfere?

Finite Sampling Ambiguity

- Sample at $t = 0, 2, 4, 6, 8,$: $\quad y \equiv 0$
- Sample at $t = 0, \frac{12}{10}, \frac{4}{3}, \ldots$ (•): $\quad \sin(\pi t/2) = \sin(2\pi t)$
- Finite sample $\Rightarrow$ high-ω “between the cracks”

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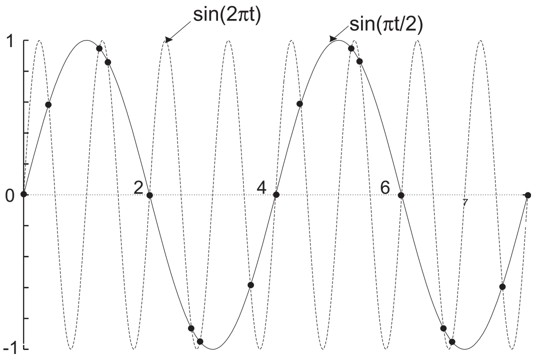

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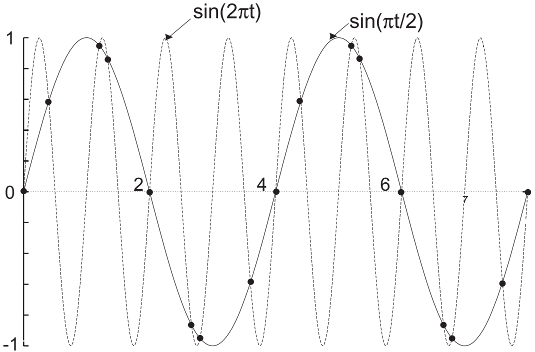

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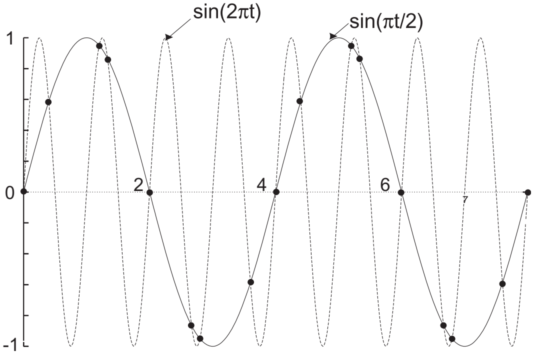

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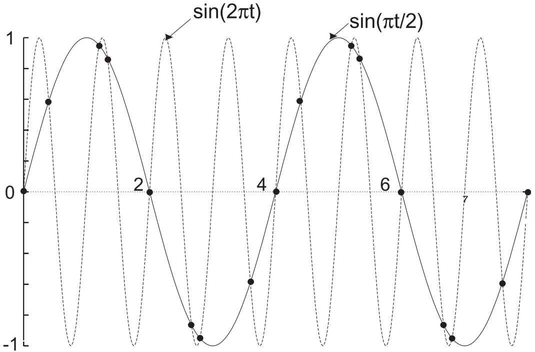

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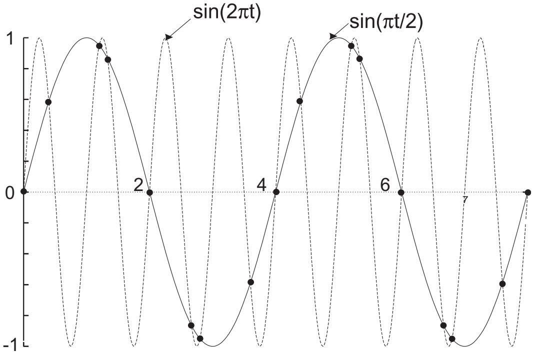

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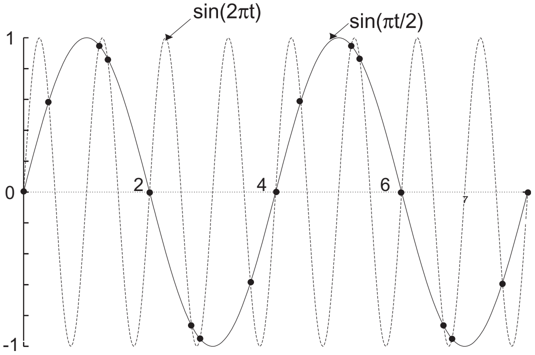

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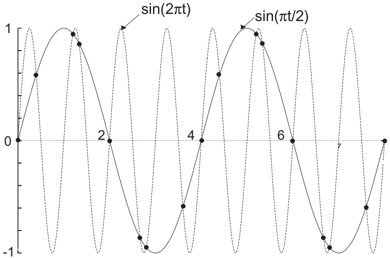

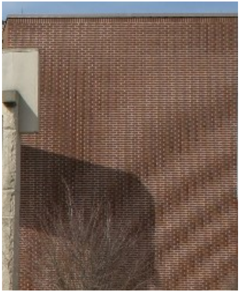

(Wikipedia)

- High-ω contaminates low
- Moiré distortion in synthesis
- "High-ω aliased by low"
- Math: for sampling rate $s = N/T$

- ω, $\omega - 2s$
- Same DFT if

$$s = \frac{N}{T} \leq \frac{\omega}{2} \qquad (1)$$

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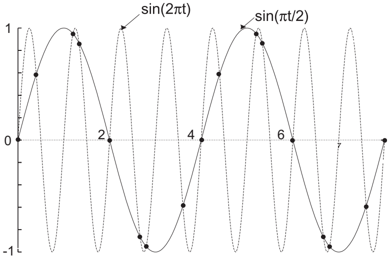

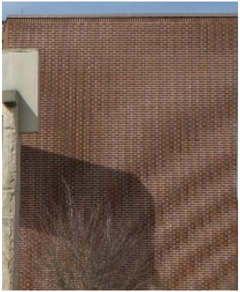
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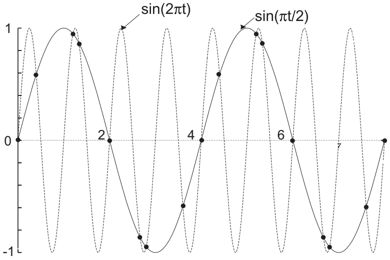

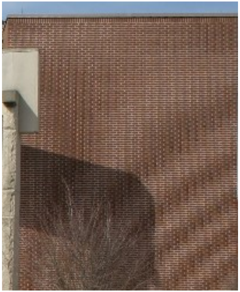

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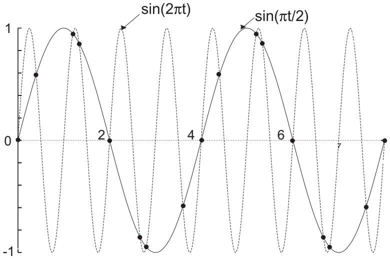

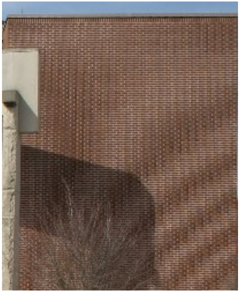

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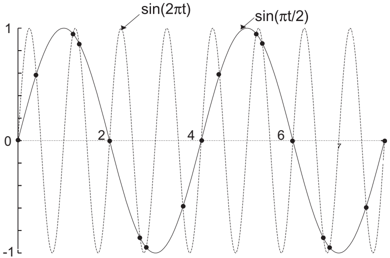

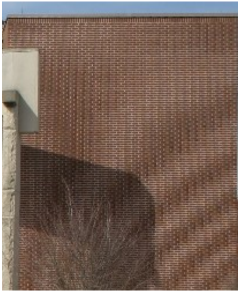

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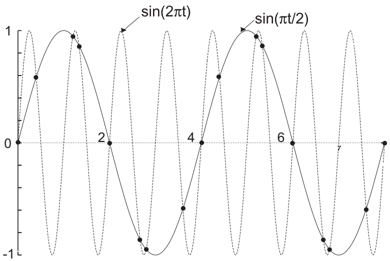

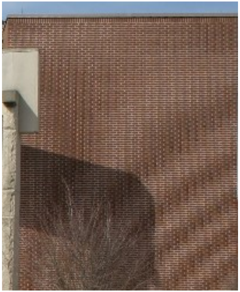

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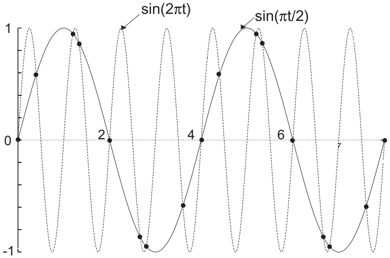

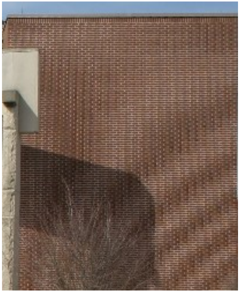

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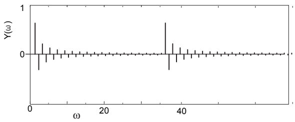
Eliminating Aliasing

- Recall: $s = N/T$ = sampling rate
- Nyquist criterion: no frequency $> s/2$ in input signal
- Filter out high ω (*e.g.* sinc filter) $\rightarrow$ good low ω

Good High ω

- Can't do high-ω right @ this sampling rate
- Need more sampling, higher s
- $\rightarrow$ higher ω in spectrum middle (ends = error prone)
- Recall: padding with 0s (larger T) $\rightarrow$ smoother $Y(\omega)$



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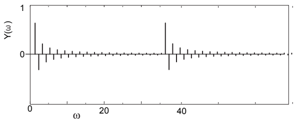

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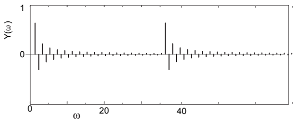

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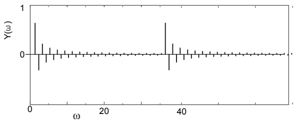

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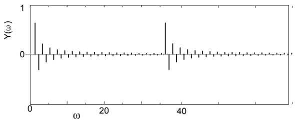

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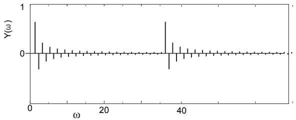

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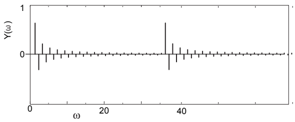

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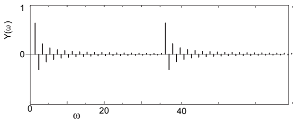

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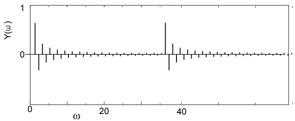

Assessment of Aliasing

1. Perform DFT on $y(t) = \sin\left(\frac{\pi}{2}t\right) + \sin(2\pi t)$.
2. True TF peaks at $\omega = \pi/2$ & $\omega = 2\pi$.
3. Look for aliasing at low sample rate.
4. Verify that aliasing vanishes at high sampling rate.
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Summary

- If sampling rate is low, some high frequency components can contaminate the deduced low-frequency components.
- The reconstructed signal will show distortions.
- Nyquist criterion to eliminate aliasing: no frequency $> (N/T)/2$ in input signal.