Computational Fluid Dynamics (CFD, CHD)*

PDE (Shocks 1st); Part I: Basics, Part II: Vorticity Fields

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Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

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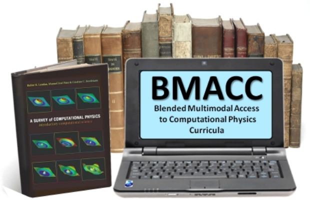
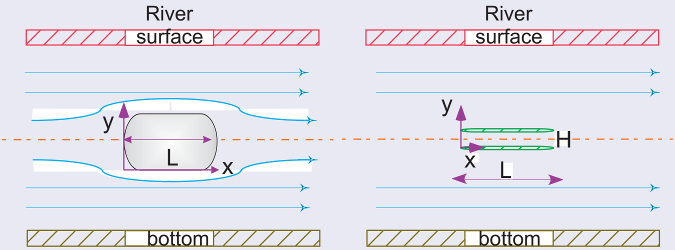
Course: **Computational Physics II**



Problem: Placement of Boulders for Migrating Salmon

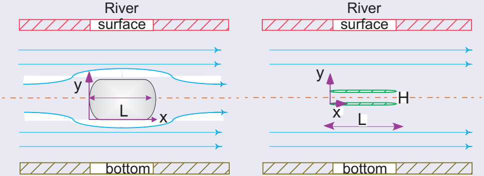
Wake Block "Force" of River?



- Deep, wide, fast-flowing streams
- "Boulder" = long rectangular beam, plates
- Objects not disturb surface/bottom flow
- Problem: large enough wake for 1m salmon

Theory: Hydrodynamics

Assumptions; Continuity Equation



$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = 0 \qquad (1)$$

$$\mathbf{j} \stackrel{\text{def}}{=} \rho\, \mathbf{v}(\mathbf{x}, t) \qquad (2)$$

- (1): Continuity equation
- 1st eqtn hydrodynamics
- Incompressible fluid
- $\Rightarrow \rho =$ constant
- Friction (viscosity)
- Steady state, $v \neq v(t)$

Navier–Stokes: 2nd Hydrodynamic Equation

Hydrodynamic Time Derivative

$$\frac{D\mathbf{v}}{Dt} \stackrel{\text{def}}{=} (\mathbf{v}\cdot\vec{\nabla})\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \quad (1)$$

- For quantity within moving fluid
- Rate of change wrt stationary frame
- Velocity of material in fluid element
- Change due to motion + explicit t dependence
- $D\mathbf{v}/Dt$: 2nd O v $\Rightarrow$ nonlinearities
- $\sim$ Fictitious (inertial) forces
- Fluid's rest frame accelerates

Now Really the Navier–Stokes Equation

Transport Fluid Momentum Due to Forces & Flow

$$\frac{D\mathbf{v}}{Dt} = \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \vec{\nabla} P(\rho, T, x) \qquad \text{(Vector Form)} \qquad (1)$$

$$\frac{\partial v_x}{\partial t} + \sum_{j=x}^{z} v_j \frac{\partial v_x}{\partial x_j} = \nu \sum_{j=x}^{z} \frac{\partial^2 v_x}{\partial x_j^2} - \frac{1}{\rho}\frac{\partial P}{\partial x} \qquad \text{(x component)} \qquad (2)$$

- ν = viscosity, P = pressure
- Recall $d\mathbf{p}/dt = \mathbf{F}$
- $D\mathbf{v}/Dt \stackrel{\text{def}}{=} (\mathbf{v} \cdot \vec{\nabla})\mathbf{v} + \partial \mathbf{v}/\partial t$
- $\mathbf{v} \cdot \nabla \mathbf{v}$: transport via flow
- $\mathbf{v} \cdot \nabla \mathbf{v}$: advection
- $\vec{\nabla} P$:change due to ΔP
- $\nu \nabla^2 \mathbf{v}$: due to viscosity
- $P(\rho, T, x)$: equation state
- Assume = $P(x)$
- Steady-state $\Rightarrow \partial_t v_i = 0$
- Incompressible $\Rightarrow \partial_t \rho = 0$

Resulting Hydrodynamic Equations

Assumed: Steady State, Incompressible, $P = P(x)$

$$\vec{\nabla} \cdot \mathbf{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i} = 0 \quad \text{(Continuity)} \tag{1}$$

$$(\mathbf{v} \cdot \vec{\nabla})\mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \vec{\nabla} P \quad \text{(Navier–Stokes)} \tag{2}$$

- (1) Continuity equation: Incompressibility, in = out
- Stream width $\gg$ beam z dimension $\Rightarrow \partial_z v \simeq 0 \Rightarrow$

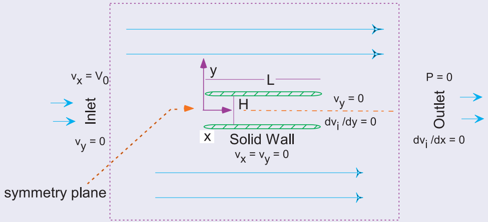
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3}$$

$$\nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) = v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} \tag{4}$$

$$\nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) = v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial y} \tag{5}$$

Boundary Conditions for Parallel Plates

Physics Determines BC $\Rightarrow$ Unique Solution

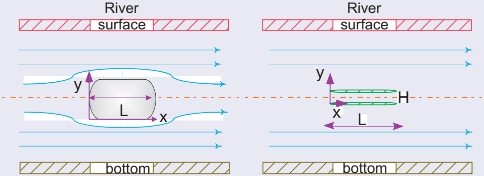


- Constant stream velocity +
- Low V_0, high viscosity $\Rightarrow$
- Laminar: smooth, no cross
- $\Rightarrow$ streamlines of motion
- Thin plates $\Rightarrow$ laminar flow

- Upstream unaffected
- Solve rectangular region
- $L, H \ll R_{stream}$ $\Rightarrow$ uniform down
- Far top, bot $\Rightarrow$ symmetry

Analytic Solution for Parallel Plates (See Text)

Bernoulli Effect: Pressure Drop Through Plates



$$v_x(y) = \frac{1}{2\rho\nu}\frac{\partial P}{\partial x}(y^2 - yH) \quad (1)$$

$$\frac{\partial P}{\partial x} = \text{known constant} \quad (2)$$

$$V_0 = 1\,\text{m/s}, \rho = 1\,\text{kg/m}^3, \nu = 1\,\text{m}^2/\text{s}, H = 1\,\text{m} \quad (3)$$

$$\Rightarrow \frac{\partial P}{\partial x} = -12, \qquad v_x(y) = 6y(1-y) \quad (4)$$

Finite-Difference Navier–Stokes Algorithm + SOR

Rectangular grid $x = ih, \quad y = jh$

- 3 Simultaneous equations $\rightarrow$ 2 ($v^y \equiv 0$)

$$v^x_{i+1,j} - v^x_{i-1,j} + v^y_{i,j+1} - v^y_{i,j-1} = 0 \tag{1}$$

$$v^x_{i+1,j} + v^x_{i-1,j} + v^x_{i,j+1} + v^x_{i,j-1} - 4v^x_{i,j} \tag{2}$$

$$= \frac{h}{2} v^x_{i,j} \left[v^x_{i+1,j} - v^x_{i-1,j} \right] + \frac{h}{2} v^y_{i,j} \left[v^x_{i,j+1} - v^x_{i,j-1} \right] + \frac{h}{2} \left[P_{i+1,j} - P_{i-1,j} \right]$$

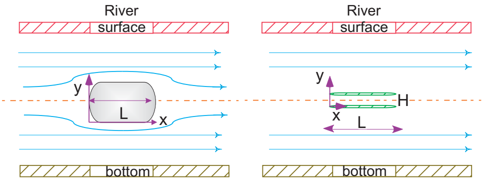
- Rearrange as algorithm for Successive Over Relaxation

$$4v^x_{i,j} = v^x_{i+1,j} + v^x_{i-1,j} + v^x_{i,j+1} + v^x_{i,j-1} - \frac{h}{2} v^x_{i,j} \left[v^x_{i+1,j} - v^x_{i-1,j} \right]$$

$$- \frac{h}{2} v^y_{i,j} \left[v^x_{i,j+1} - v^x_{i,j-1} \right] - \frac{h}{2} \left[P_{i+1,j} - P_{i-1,j} \right] \tag{3}$$

- Accelerate convergence + SOR; $\omega > 2$ unstable

End Part I: Basics



Part II: Vorticity Form of Navier–Stokes Equation

2 HD Equations in Terms of Stream Function $\mathbf{u}(\mathbf{x})$

$$\vec{\nabla} \cdot \mathbf{v} = 0 \qquad \text{Continuity} \tag{1}$$

$$(\mathbf{v} \cdot \vec{\nabla})\mathbf{v} = -\frac{1}{\rho}\vec{\nabla} P + \nu \nabla^2 \mathbf{v} \qquad \text{Navier–Stokes} \tag{2}$$

- Like EM, simpler via (scalar & vector) potentials
- Irrotational Flow: no turbulence, scalar potential
- Rotational Flow: 2 vector potentials; 1st stream function

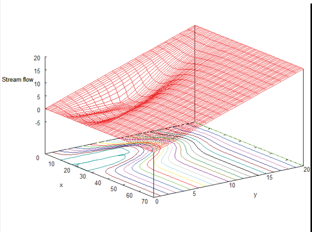
$$\mathbf{v} \stackrel{\text{def}}{=} \vec{\nabla} \times \mathbf{u}(\mathbf{x}) \tag{3}$$

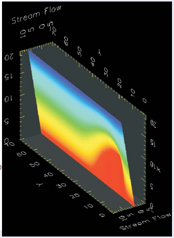
$$= \hat{\epsilon}_x \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) + \hat{\epsilon}_y \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \tag{4}$$

- $\vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{u}) \equiv 0 \;\Rightarrow$ automatic continuity equation

2 HD Equations in Terms of Stream Function (cont)

2-D flow: $u =$ Constant Contour Lines = Streamlines





$$\mathbf{v} \stackrel{\text{def}}{=} \vec{\nabla} \times \mathbf{u}(\mathbf{x}) \tag{1}$$

$$= \hat{\epsilon_x} \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) + \hat{\epsilon_y} \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \tag{2}$$

$$\mathbf{v}_z = 0 \Rightarrow \mathbf{u}(\mathbf{x}) = u\hat{\epsilon_z} \tag{3}$$

$$\Rightarrow \quad v_x = \frac{\partial u}{\partial y}, \quad v_y = -\frac{\partial u}{\partial x} \tag{4}$$

Introduce Vorticity $\mathbf{w}(\mathbf{x}) \sim \vec{\omega}$

Vortex: Spinning, Often Turbulent Fluid Flow

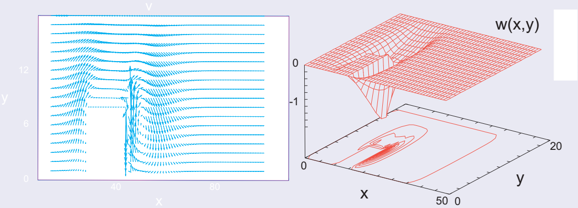
$$\mathbf{w} \stackrel{\text{def}}{=} \vec{\nabla} \times \mathbf{v}(\mathbf{x}) \quad (1)$$

$$w_z = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \quad (2)$$

- Measure of $\vec{v}$'s rotation
- RH rule fluid element
- $\mathbf{w} = 0 \;\Rightarrow$ irrotational
- $\mathbf{w} = 0 \;\Rightarrow$ uniform
- Moving field lines
- Relate to stream function:

Introduce Vorticity $\mathbf{w}(\mathbf{x}) \sim \vec{\omega}$

$\sim$ Poisson's equation $\nabla^2\phi = -4\pi\rho$



$$\mathbf{w} \stackrel{\text{def}}{=} \vec{\nabla} \times \mathbf{v}(\mathbf{x}) \quad (1)$$

$$\mathbf{w} = \vec{\nabla} \times \mathbf{v} = \vec{\nabla} \times (\vec{\nabla} \times \mathbf{u}) = \vec{\nabla}(\vec{\nabla} \cdot \mathbf{u}) - \nabla^2\mathbf{u} \quad (2)$$

$$\text{yet} \quad \mathbf{u} = u(x,y)\hat{\epsilon}_z \Rightarrow \vec{\nabla} \cdot \mathbf{u} = 0 \quad (3)$$

$$\Rightarrow \vec{\nabla}^2\mathbf{u} = -\mathbf{w} \quad (4)$$

- Like Poisson with ea **w** component = source

Vorticity Form of Navier–Stokes Equation

Take Curl of Velocity Form

$$\vec{\nabla} \times \left[(\mathbf{v} \cdot \vec{\nabla})\mathbf{v} \;=\; \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \vec{\nabla} P \quad \text{(Navier–Stokes)} \right] \tag{1}$$

$$\nu \nabla^2 \mathbf{w} \;= [(\vec{\nabla} \times \mathbf{u}) \cdot \vec{\nabla}]\mathbf{w} \tag{2}$$

- In 2-D + only z components:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; -\, w \tag{3}$$

$$\nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \;=\; \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial w}{\partial y} \tag{4}$$

- Simultaneous, nonlinear, elliptic PDEs for u & w
- $\sim$ Poisson's + wave equation + friction + variable ρ

Relaxation Algorithm (SOR) for Vorticity Equations

$x = ih, \quad y = jh$

- CD Laplacians, 1st derivatives

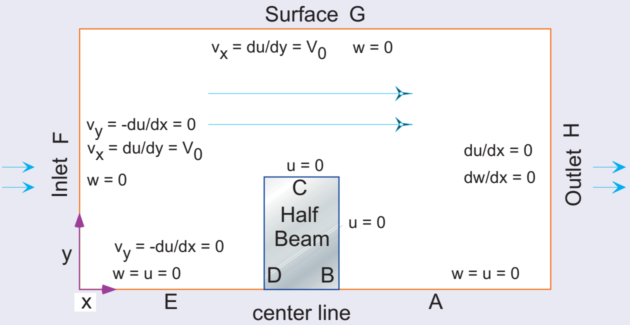
$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + h^2 w_{i,j}\right) \qquad (1)$$

$$w_{i,j} = \frac{1}{4}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) - \frac{R}{16}\{[u_{i,j+1} - u_{i,j-1}] \times [w_{i+1,j} - w_{i-1,j}] - [u_{i+1,j} - u_{i-1,j}]\,[w_{i,j+1} - w_{i,j-1}]\} \qquad (2)$$

$$R = \frac{1}{\nu} = \frac{V_0 h}{\nu} \quad \text{(in normal units)} \qquad (3)$$

- R = grid Reynolds number ($h \to R_{pipe}$); measure nonlinear
- Small R: smooth flow, friction damps fluctuations
- Large R ($\simeq 2000$): laminar $\to$ turbulent flow
- Onset of turbulence: hard to simulate (need kick)

Boundary Conditions for Beam



- Well-defined solution of elliptic PDEs requires *u*, *w* BC
- Assume inlet, outlet, surface far from beam
- Freeflow: No beam
- NB $w = 0 \Rightarrow$ no rotation
- Symmetry: identical flow above, below centerline, not thru

Boundary Conditions for Beam (cont)

See Text for More Explanations

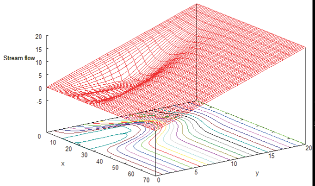
- Centerline: = streamline, $u =$ const =0 (no $v_\perp$
- No flow in, out beam to it $\Rightarrow u = 0$ all beam surfaces
- Symmetry $\Rightarrow$ vorticity $w = 0$ along centerline
- Inlet: horizontal fluid flow, $v = v_x = V_0$:
- Surface: Undisturbed $\Rightarrow$ free-flow conditions:
- Outlet: Matters little; convenient choice: $\partial_x u = \partial_x w$
- Beamsides: $v_\perp = u = 0$; viscous $\Rightarrow v_\parallel = 0$
- Yet, over specify BC $\Rightarrow$ only no-slip vorticity w:
- Viscosity $\Rightarrow v_x = \frac{\partial u}{\partial y} = 0$ (beam top)
- Smooth flow on beam top $\Rightarrow v_y = 0$ + no x variation:

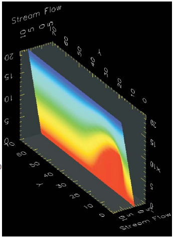
$$\frac{\partial v_y}{\partial x} = 0 \Rightarrow w = -\frac{\partial v_x}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \tag{1}$$

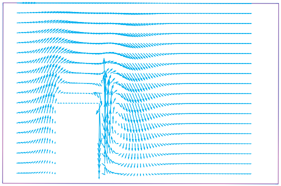
Implementation & Assessment:SOR on a Grid

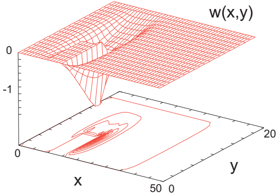
- Basic soltn vorticity form Navier–Stokes: `Beam.py`
- NB relaxation = simple, BC $\neq$ simple
- Separate relaxation of stream function & vorticity
- Explore convergence of up & downstream u
- Determine number iterations for 3 place with $\omega = 0, 0.3$
- Change beam's horizontal position so see wave develop
- Make surface plots of u, w, $\mathbf{v}$ with contours; explain
- Is there a resting place for salmon?

Results


Stream flow
20
15
10
5
0
-5
0
10
20
30
40
50
60
70
x
0
5
10
15
20
y


Stream Flow
Stream Flow




w(x,y)
0
-1
0
x
50
0
y
20