Solving PDEs for Electrostatics

Via Relaxation (Simple, Not Industrial Strength)

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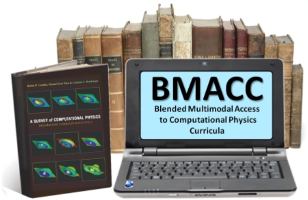
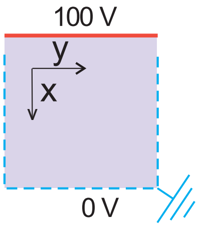
Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

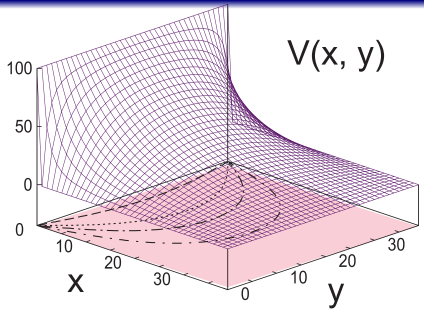
with Support from the National Science Foundation

Course: **Computational Physics II**



Problem: V for Arbitrary Geometry & BCs





Solve Inside Charge-Free Square!

- Assume conductor @ V_{fixed} = simulation region
- Closed boundary (insulate openings)
- $\Rightarrow$ Neumann conditions on the boundary
- $\Rightarrow$ unique & stable solution

Laplace & Poisson Elliptic PDEs (Theory)

- Classical EM, static charges, Poisson's Equation:

$$\nabla^2 U(\mathbf{x}) = -4\pi\rho(\mathbf{x})$$

- Laplace's equation if $\rho(\mathbf{x}) = 0$:

$$\nabla^2 U(\mathbf{x}) = 0$$

- Solve in 2-D rectangular coordinates:

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = \begin{cases} 0, & \text{Laplace's equation,} \\ -4\pi\rho(\mathbf{x}), & \text{Poisson's equation} \end{cases}$$

- $U(x,y)$: two independent variables $\Rightarrow$ PDE
- Laplace's: charge indirectly generate BC

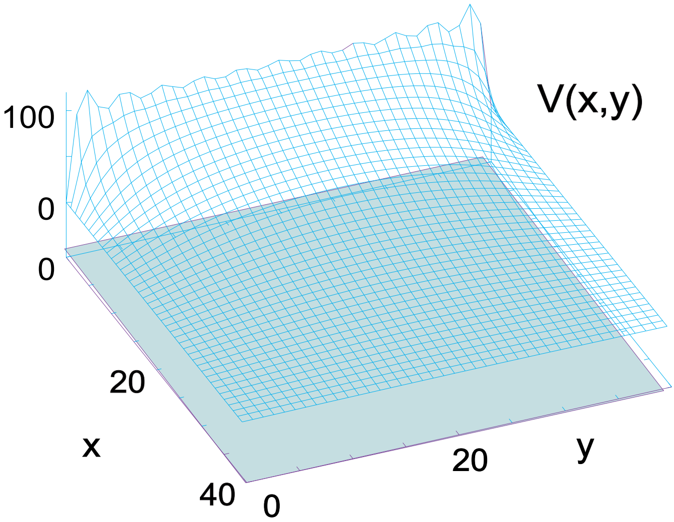
Fourier Series Solution As Algorithm

Standard Textbook Not Always Good

$$U(x, y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{400}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh(n\pi y/L)}{\sinh(n\pi)}$$

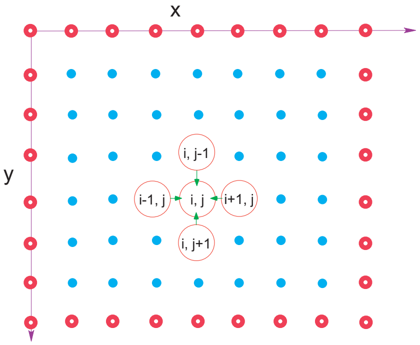
- Sum not separable: $\neq \; X(x)\ Y(y)$
- Sum = infinite; not true analytic solution
- Algorithm: $\sum^{\infty} \simeq \sum^{N}$
- Painfully slow convergence $\Rightarrow$ round-off error
- $\sinh(n)$ overflow for large n:
- 40,000 terms Fourier *vs* 200 algorithm
- Converges in *mean square*, Gibbs overshoot $N \neq \infty$

Fourier- Gibb's Overshoot at Discontinuities



Finite-Difference Form of Poisson Equation

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = \begin{cases} 0, & \text{Laplace's equation,} \\ -4\pi\rho(\mathbf{x}), & \text{Poisson's equation} \end{cases}$$



- Form 2-D $x - y$ lattice
- Solve U each lattice site
- Derivatives = finite-differences
- Finite-elements matches small geometric elements
- Elements; more efficient, harder setup

Finite-Difference Form of Poisson Equation

$$U(x+\Delta x, y) = U(x,y) + \frac{\partial U}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(\Delta x)^2 + \cdots$$

$$U(x-\Delta x, y) = U(x,y) - \frac{\partial U}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(\Delta x)^2 - \cdots$$

1. FD $\partial/\partial x$
2. Add R, L series:
3. Odd terms cancel:

$$\frac{\partial^2 U(x,y)}{\partial x^2} \simeq \frac{U(x+\Delta x, y) + U(x-\Delta x, y) - 2U(x,y)}{(\Delta x)^2}$$

$\Rightarrow$ Finite-difference Poisson PDE:

$$\frac{U(x+\Delta x, y) + U(x-\Delta x, y) - 2U(x,y)}{(\Delta x)^2} + \frac{U(x, y+\Delta y) + U(x, y-\Delta y) - 2U(x,y)}{(\Delta y)^2} = -4\pi\rho$$

Solve Poisson Equation on Lattice $x = i\Delta$, $y = j\Delta$

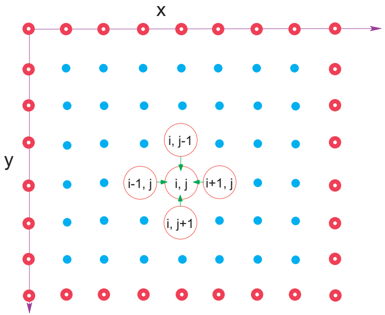
After break

Solve Discrete Poisson Equation on Lattice

$$-4\pi\rho(\mathbf{x}) = \frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} \quad (1)$$

$$-4\pi\rho_{i,j} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} \quad (2)$$

$$\Rightarrow \quad U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right] + \pi\rho_{i,j}\Delta^2 \quad (3)$$



- Solve (2) big matrix
- Correct solution = average 4 nearest neighbors
- Iteration: BC → solution
- Relax to solution
- Know if arrive or fail