Electromagnetic Waves

The Finite-Difference Time Domain (FDTD) Algorithm

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Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

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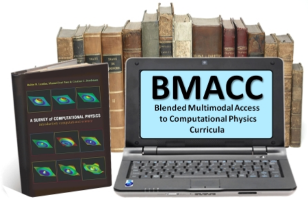
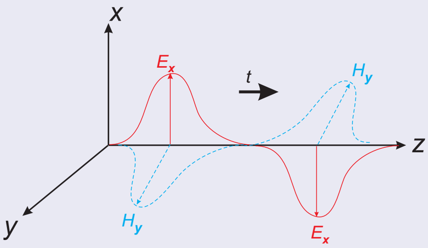
Course: **Computational Physics II**



Problem: Determine E & H Fields for All Times

Given: Space $0 \leq z \leq 200$

$$E_x(z, t=0) \;=\; 0.1 \sin \frac{2\pi z}{100}, \qquad H_y(z, t=0) \;=\; 0.1 \sin \frac{2\pi z}{100}$$



- E&M practical import
- FDTD Δz, Δt = step
- New: coupled fields
- New: vector fields, 3-D

Theory: Maxwell's Equations in Free Space

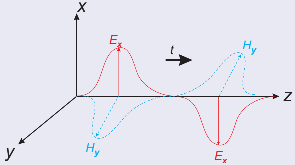
$\mathbf{E} = E_x$, $\mathbf{H} = H_y$, $\mathbf{S} = \mathbf{E} \times \mathbf{H} = S_z \Rightarrow$ 3-D

$$\vec{\nabla} \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x(z,t)}{\partial x} = 0 \quad \text{(Tranverse)} \qquad (1)$$

$$\vec{\nabla} \cdot \mathbf{H} = 0 \quad \Rightarrow \quad \frac{\partial H_y(z,t)}{\partial y} = 0 \quad \text{(Tranverse)} \qquad (2)$$

$$\frac{\partial \mathbf{E}}{\partial t} = +\frac{1}{\epsilon_0} \vec{\nabla} \times \mathbf{H} \quad \Rightarrow \quad \frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y(z,t)}{\partial z} \qquad (3)$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0} \vec{\nabla} \times \mathbf{E} \quad \Rightarrow \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x(z,t)}{\partial z} \qquad (4)$$



Finite Difference Time Domain (FDTD) Algorithm

Central-Difference Derivatives $\Rightarrow$

$$E_x^{z,t} = E_x^{k,n+1/2}, \quad H_y^{z,t} = H_y^{k+1/2,n}$$

$$\frac{\partial E(z,t)}{\partial t} \simeq \frac{E(z, t+\frac{\Delta t}{2}) - E(z, t-\frac{\Delta t}{2})}{\Delta t}, \tag{1}$$

$$\frac{\partial E(z,t)}{\partial z} \simeq \frac{E(z+\frac{\Delta z}{2}, t) - E(z-\frac{\Delta z}{2}, t)}{\Delta z} \tag{2}$$

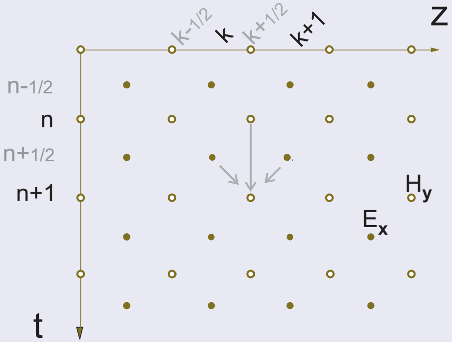
Substitute into Maxwell, rearrange for t stepping

$$E_x^{k,n+1/2} = E_x^{k,n-1/2} - \frac{\Delta t}{\epsilon_0\,\Delta z}\left(H_y^{k+1/2,n} - H_y^{k-1/2,n}\right), \tag{3}$$

$$H_y^{k+1/2,n+1} = H_y^{k+1/2,n} - \frac{\Delta t}{\mu_0 \Delta z}\left(E_x^{k+1,n+1/2} - E_x^{k,n+1/2}\right) \tag{4}$$

Displaced E_x, H_y Space-Time Lattices

$$E_x^{z,t} = E_x^{k,n+1/2}, \quad H_y^{z,t} = H_y^{k+1/2,n}$$



- Space variation H_y $\Rightarrow$ time variation E_x
- Space variation E_x $\Rightarrow$ time variation H_y

Alternate Formulation: Even & Odd Times

Double Index Values

$$E_x^{k,n} = E_x^{k,n-2} - \frac{\Delta t}{\epsilon_0 \Delta z}\left(H_y^{k+1,n-1} - H_y^{k-1,n-1}\right), \qquad k \text{ even}, \ \text{odd}, \tag{1}$$

$$H_y^{k,n} = H_y^{k,n-2} - \frac{\Delta t}{\mu_0 \Delta z}\left(E_x^{k+1,n-1} - E_x^{k-1,n-1}\right), \qquad k \text{ odd}, \ \text{even}. \tag{2}$$

- E: even z, odd t
- H: odd z, even t

Normalized Algorithm; Stability Analysis

$\tilde{E}$ With Same Dimension as H, $\tilde{E} = \sqrt{\epsilon_0/\mu_0}\,E$

$$\tilde{E}_x^{k,n+1/2} = \tilde{E}_x^{k,n-1/2} + \beta\left(H_y^{k-1/2,n} - H_y^{k+1/2,n}\right) \quad (1)$$

$$H_y^{k+1/2,n+1} = H_y^{k+1/2,n} + \beta\left(\tilde{E}_x^{k,n+1/2} - \tilde{E}_x^{k+1,n+1/2}\right) \quad (2)$$

$$\beta = \frac{c}{\Delta z/\Delta t}, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \text{(light)} \quad (3)$$

- $\beta =$ light/grid speed
- $\omega_{wave} \Rightarrow$ t scale
- $\lambda_{wave} \Rightarrow$ z scale
- $>$ 10 points/λ

- Courant Stability: $\beta \leq 1/2$
- Smaller $\Delta t \uparrow$ precision
- Smaller $\Delta t \uparrow$ stability
- Smaller $\Delta z \Rightarrow$ smaller Δt

Implementation **FDTD.py**

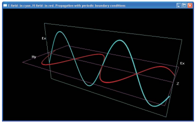
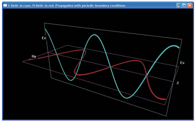
- Initial conditions ($0 \leq z(k) \leq 200$):

$$E_x(z, t=0) = 0.1 \sin \frac{2\pi z}{100}, \qquad H_y(z, t=0) = 0.1 \sin \frac{2\pi z}{100}$$

- Discrete Maxwell equations:

$$Ex[k,1] = Ex[k,0] + beta * (Hy[k-1,0] - Hy[k+1,0])$$
$$Hy[k,1] = Hy[k,0] + beta * (Ex[k-1,0] - Ex[k+1,0])$$

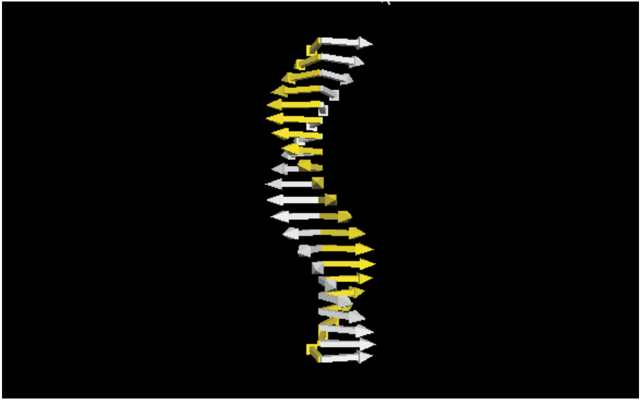
- 0 = old time, 1 = new time
- Spatial endpoints via periodic boundary conditions:

Assessment

1. Impose BC such that fields vanish on boundaries
2. Show effect of these BCs
3. Test Courant stability condition
4. Solve with inserted dielectric slab
5. Note transmission, reflection at slab boundaries
6. Verify that $\mathbf{H}(\mathbf{t} = \mathbf{0}) = 0 \;\Rightarrow$ right & left pulses
7. Investigate resonator modes for plane waves with nodes at boundaries

Extension: Circularly Polarized Waves



CircPolartzn.py