Computational Fourier Analysis

Mathematics, Computing and Nonlinear Oscillations

Rubin H Landau

Sally Haerer, Producer-Director

Oregon State University
with National Science Foundation Support

Course: **Computational Physics I**

(Prerequisite: *Intro Computational Science*)

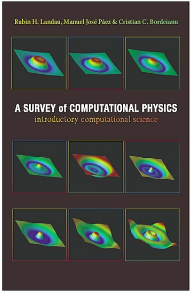
BMACC
Blended Multimodal Access to Computational Physics Curricula

Outline

Text:

Three Fourier Units

Unit I:	Fourier series: Review & DFT (Only part of I today!)
Unit II:	Signal filtering to reduce noise
Unit III:	Fast Fourier transform (FFT) $\Rightarrow$ real time DFT

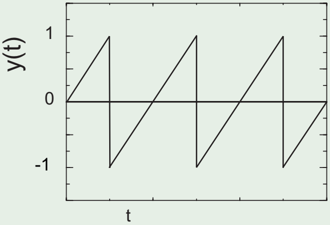


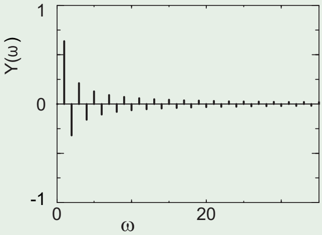
Today's Lecture:

Problem: Amount of $\sin \omega t$ in Nonlinear Oscillation?

Example

- Example (in green): Sawtooth function





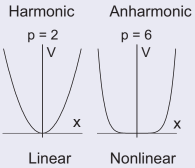
- What "frequencies" present? (What does this mean?)
- Meaning: "Fourier decomposition (right)"
- Periodic, many sinusoids; yet, one period
- Only steady state, let transients die out

Physics Problem (see ODE): Nonharmonic Oscillator

Nonharmonic (nontranditional) oscillator

$$V(x) = k|x|^p, \quad p \neq 2$$

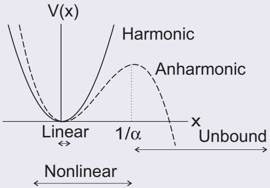
- large $p \Rightarrow$ square well
- square well $\Rightarrow$ sawtooth



Perturbed oscillator

$$V(x) = \frac{1}{2}kx^2\left(1 - \frac{2}{3}\alpha x\right)$$

- Linear: 1st approximation
- All Vs $\rightarrow$ constant period T
- Periodic not sinusoidal



Math (Theory): Fourier Series

Fourier's Theorem: Function Representation

- Single-valued periodic function ("signal")
- Finite number discontinuities

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \qquad (1)$$

- $a_n \leftrightarrow \cos n\omega t$ "in" $y(t)$
- Intensity(ω), Power(ω) $\propto a_n^2 + b_n^2$
- True frequency $\omega \equiv \omega_1 = \frac{2\pi}{T} \neq$ HO frequency
- Need know period T, $y(t + T) = y(t)$
- Period may depend on amplitude

Math: Fourier Series Fit

Fourier Series

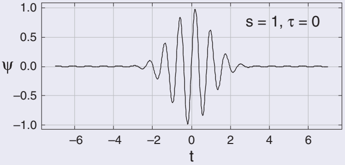
$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad (2)$$

Theorem: Series = "Best Fit"

- Least-squares sense: minimizes $\sum_i [y(t_i) - y_i]^2$
- $\Rightarrow \sum_i^{\infty} a_n \cos n\omega t \simeq \langle y(t) \rangle$
- $\Rightarrow$ miss discontinuities
- Need infinite number of terms
- Not exact math fit (power series)
- Not good numerical solution (∞ terms)
- Not closed form analytic solution

Fourier Integrals (Transforms)

Nonperiodic Functions



$y(t) =$
$b_0 + b_1 \cos \omega_1 t + b_2 \cos 2\omega_1 t + \cdots$

- Nonperiodic $\equiv T \to \infty$
- $\Rightarrow$ continuous ω
- Math: $\sum \to \int$
- Numerically: $\int \to \sum$
- Numerically: the same!

Fourier Semantics

- $\omega_1 = \frac{2\pi}{T}$ = fundamental
- $n\omega_1$ = nth harmonic
- $\omega > \omega_1$: overtones
- b_0: DC component
- $\sum$ normal modes
- Modes: see next slide

Fourier Series for Nonlinear Oscillations?

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \qquad (3)$$

OK by Theorem if periodic, but

- Nonlinear oscillators: $\omega_1 = \omega(A) = 2\pi/T$
- Individual terms $\neq$ solution
- Linear eqtn $\leftrightarrow$ principle of linear superposition

$$y_1(t), y_2(t) = \text{solutions} \qquad (4)$$

$$\Rightarrow \alpha y_1(t) + \beta y_2(t) = \textit{solution} \qquad (5)$$

- Useful if not too nonlinear
- Broadband spectrum $\Rightarrow$ chaos
- Many strong overtones

Determine Fourier Coefficients $a_n, \ b_n$

Project Out Components

$$\langle \omega | y \rangle = \int_0^{2\pi} dt \cos n\omega t \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \right] \tag{6}$$

$$\Rightarrow \quad \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{T} \int_0^T dt \begin{pmatrix} \cos n\omega t \\ \sin n\omega t \end{pmatrix} y(t) \tag{7}$$

- $a_0 = 2 \langle y(t) \rangle$
- Be smart: use Symmetry
- Odd $y(-t) = -y(t) \ \Rightarrow \ a_n \equiv 0$
- Even $y(-t) = y(t) \ \Rightarrow \ b_n \equiv 0$
- Realistic calculations: $\int \to \sum, \ y(t < 0) = ? \quad \Rightarrow \quad$ small b_n

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Project Out Components

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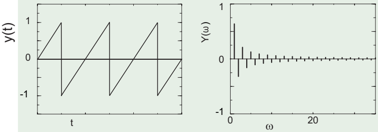
Example: Sawtooth Function, Analytic a_n, b_n

Example



$$y(t) = \begin{cases} \frac{t}{T/2}, & 0 \le t \le \frac{T}{2} \\ \frac{t-T}{T/2}, & \frac{T}{2} \le t \le T \end{cases} \quad (7)$$

- Periodic, nonharmonic, discontinuous, sharp corners
- Odd ($\Rightarrow$ sine series), obvious shift to left:

$$y(t) = \frac{t}{T/2}, \qquad -\frac{T}{2} \le t \le \frac{T}{2} \quad (8)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} dt\, \sin n\omega t \frac{t}{T/2} \quad \text{(above)} \quad (9)$$

$$\Rightarrow \quad y(t) = \frac{2}{\pi}\left[\sin \omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t \cdots\right] \quad (10)$$

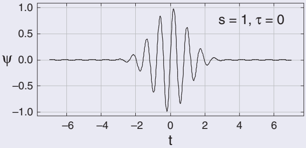
Exercise: Sawtooth Function Synthesis

1. Sum Fourier series for $n = 2, 4, 10, 20$ terms.
2. Plot two periods & compare to signal.
3. Check that series gives the mean value (0) at discontinuity
4. **Gibbs overshoot:** Check that series overshoots discontinuity by about 9%, and that this persists even when summing over a large number of terms.
5. You deserve a break right now!

Math (Theory): Fourier Transforms

Fourier series = right tool for periodic functions

Fourier Integral for Nonperiodic Functions

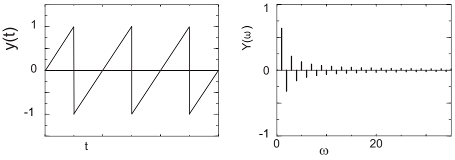


- Wave packets, pulses
- Continuous frequencies

Signal: $$y(t) = \int_{-\infty}^{+\infty} d\omega \; Y(\omega) \, \frac{e^{i\omega t}}{\sqrt{2\pi}} \tag{11}$$

- Think linear superposition: $\exp(i\omega t) = \cos\omega t + i\sin\omega t$
- Complex transform $Y(\omega) \sim (a_n, b_n) \sim$ coefficients
- $1/\sqrt{2\pi}, \; +i\omega t$ = physics conventions

Fourier Transform: $y(t) \rightarrow Y(\omega)$



Fourier Amplitudes (a_n) → Transform

$$\text{Signal, function:} \quad y(t) = \int_{-\infty}^{+\infty} d\omega \; Y(\omega) \, \frac{e^{+i\omega t}}{\sqrt{2\pi}} \tag{12}$$

$$\text{Transform, spectral:} \quad Y(\omega) = \int_{-\infty}^{+\infty} dt \; y(t) \, \frac{e^{-i\omega t}}{\sqrt{2\pi}} \tag{13}$$

- Power Spectrum $\propto |Y(\omega)|^2$, maybe $\log_{10}|Y|^2$

Math Identity: Inversion of Transform

Substitute for $y(t)$ and rearrange

$$Y(\omega) = \int_{-\infty}^{+\infty} dt \, \frac{e^{-i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' \, \frac{e^{i\omega' t}}{\sqrt{2\pi}} Y(\omega') \tag{14}$$

$$Y(\omega) = \int_{-\infty}^{+\infty} d\omega' \left\{ \int_{-\infty}^{+\infty} dt \, \frac{e^{i(\omega'-\omega)t}}{2\pi} \right\} Y(\omega') \tag{15}$$

- Depends on noncomputable Dirac delta function:

$$\int_{-\infty}^{+\infty} dt \, e^{i(\omega'-\omega)t} = 2\pi\delta(\omega' - \omega) \tag{16}$$

- Don't try on computer!

Summary

- Most any periodic function “represented” by Fourier series.
- Most any nonperiodic function “represented” by Fourier integral.
- Infinite series or integral not practical algorithm or in experiment.
- Must know actual period.
- Period depends on amplitude in general.
- DFT (next): when done is simple, elegant and powerful
- Fast Fourier transform (FFT) is faster.
- That’s all folks!