Numerical Solution of Differential Equations

- *Big topic; we'll cover in parts*
- *Not hard, very useful*
- *Context of practical problem*
- *Can browse beginning, if already known*

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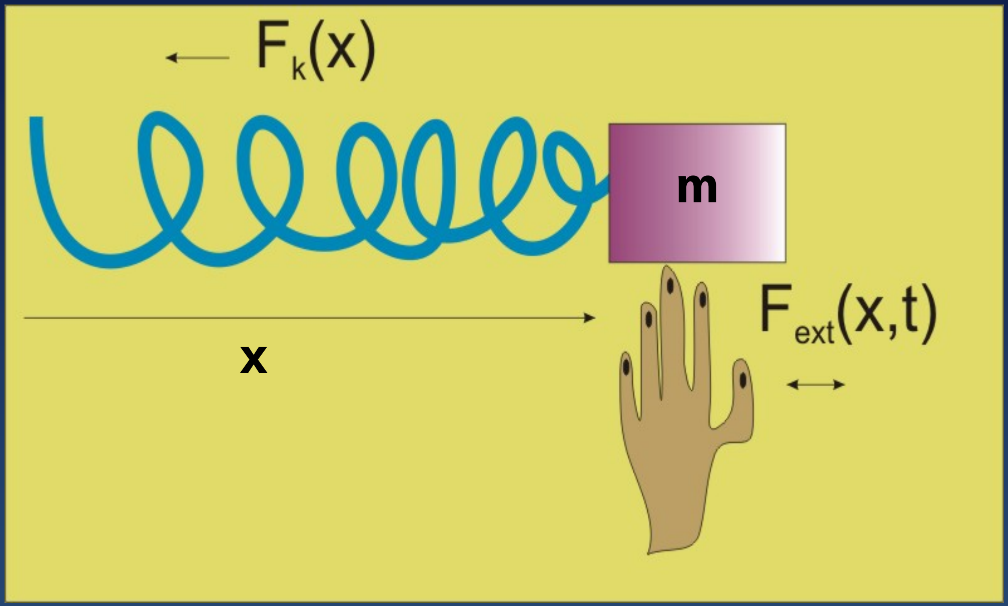
BS Degree Program: Oregon State University

Support by NSF & OSU

Problem: Forced Nonlinear Oscillator

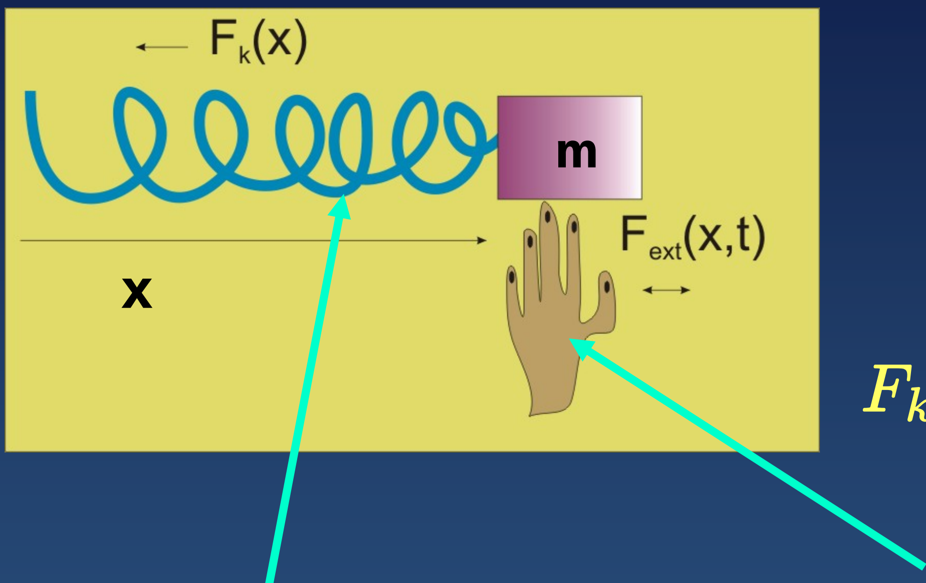


- **Mass *m*, spring, 2 forces:**
- **Spring: any *x* dependence**
- **Force: external (t), friction ...**
- ***Problem: position x(t) (1-D)***

- **Computational: arbitrary forces = easy**
- **Traditional treatments: small *x*, linear *F(x)***

Physics Theory: Newton's Laws

- **Newton's 2nd law ⇒ equation of motion *(Solve This!)***



$$\sum_i F_i = ma \qquad (1)$$

$$F_k(x) + F_{\text{ext}}(x,t) = m\frac{d^2x}{dt^2} \qquad (2)$$

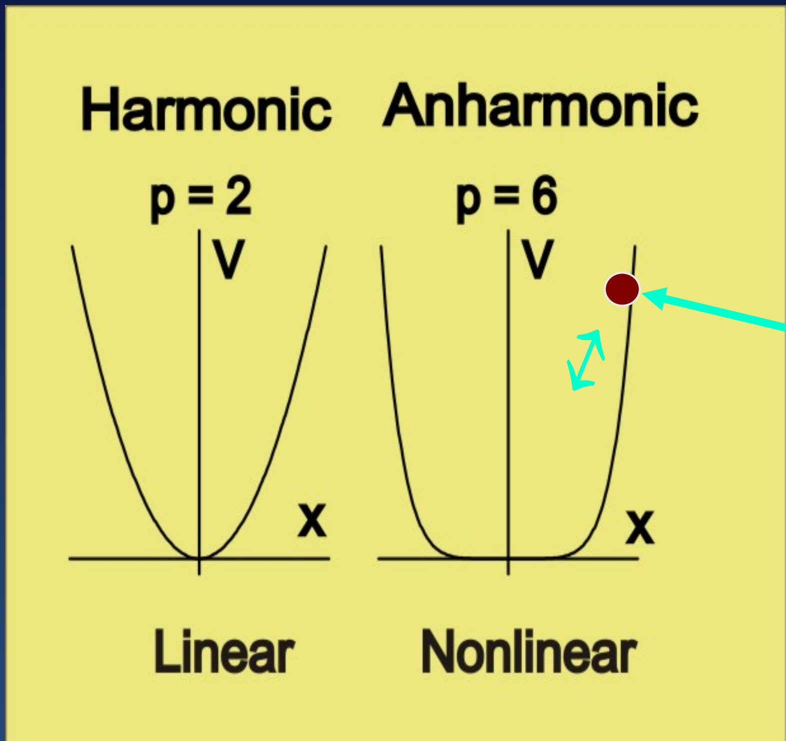
$F_k(x)$ **spring**

$F_{ext}(x,t)$ **external force**

philosphy

- **Solve (2) = Ordinary Differential Eqn = (ODE)**
- **ODE: many physical laws; nature, concrete?**
- **Integral equations (easy now): maybe more?**

Model: Nonlinear Oscillator



Potential: arbitrary power *p*

$$V(x) = \frac{1}{p} k x^p \qquad (3)$$

- **Even *p* $\Rightarrow$ restoring**
- ***p* > *6:* particle in a box**

Force on mass

$$F_k(x) = -\frac{dV(x)}{dx} = -kx^{p-1} \qquad (4)$$

- **Newton's law: 2nd -*O* ODE**

$$F_{\text{ext}}(x,t) - kx^{p-1} = m\frac{d^2x}{dt^2} \qquad (5)$$

Math: Types of Differential Equations

- **Landau's 1st Rule of education**
- **<u>Order</u> = degree of derivative**
    - **First Order ODE**

$$\frac{dy}{dt} = f(t, y) \qquad (6)$$

    - **RHS: arbitrary derivative, "force" function *f (t, y)***
        - **e.g. *f (t, y)* nonlinear in *y***

$$\frac{dy}{dt} = -3t^2y + t^9 + y^7 \qquad (7)$$

        - **Second Order ODE** (*e.g.* **Newton's law)**

$$m\frac{d^2y}{dt^2} = -3t^2\left(\frac{dy}{dt}\right)^4 + t^9y(t), \qquad (8)$$

Just letters!

- **<u>Independent</u> variable:** *t* = time**,** **Dependent** *y (t)* = position; *t (x)*

More Words: *Ordinary, Partial, Initial, Boundary*

- **Ordinary Differential Equation = ODE**
  - **only 1 independent variable:**

$$V(x)\psi(x) = -\frac{d\,\psi(x)}{dx} \qquad (9)$$

- **Partial Differential Equation =PDE (later)**
  - **> 1 independent variable:**

$$i\frac{\partial\psi(x,y,t)}{\partial t} = -\cdot\ \frac{\partial^2\,\psi(x,y,t)}{\partial x^2} + \frac{\partial^2\,\psi(x,y,t)}{\partial y^2}\Big] \qquad (10)$$

- **Initial Conditions: solve 1st-order ODE: 1 constant, $\psi(0)$**

  2nd-order: 2 constants, $\psi(0)$, $\psi'(0)$

- **Boundary Conditions**
  - **solution: fixed value(s) in space**

  - **needed for PDE (more degrees freedom)**
  - **ODE: extra restriction $\Rightarrow$ eigenvalue problem**

Words: Linear and Nonlinear ODEs

- **Nonlinear DE (dependent): *y(t)* or *dy/dt*:**

$$\frac{dy}{dt} = g^3(t)y(t), \qquad \text{linear} \tag{11}$$

$$\frac{dy}{dt} = \lambda y(t) - \lambda^2 y^2(t) \qquad \text{nonlinear} \tag{12}$$

- **Nonlinear: hard analytically; all same numerically**

- **Law <u>linear</u> superposition:**

solutions

Also a solution →

$$y(t) = \alpha\, A(t) + \beta\, B(t) \tag{13}$$

Counter e.g.

$$\frac{dy}{dt} = \lambda y(t) - \lambda^2 y^2(t) \qquad \text{(Nonlinear ODE)} \tag{14}$$

$$y(t) = \frac{a}{1 + be^{-\lambda t}} \qquad \text{(Soltn)} \tag{15}$$

$$y_1(t) = \frac{a}{1 + be^{-\lambda t}} + \frac{a'}{1 + b'e^{-\lambda t}} \qquad \text{(Not soltn)} \tag{16}$$

You deserve a break now!

Applied Math & Classical Dynamics: Standard Form of ODEs

- **All order ODEs**

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(t, \mathbf{y}). \quad (17)$$

(no ***dy*⁽ⁱ⁾**/*dt*)

N-D vectors

$$\mathbf{y} = \begin{bmatrix} y^{(0)}(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(N-1)}(t) \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f^{(0)}[t, \mathbf{y}] \\ f^{(1)}[t, \mathbf{y}] \\ \vdots \\ f^{(N-1)}[t, \mathbf{y}] \end{bmatrix}, \quad (18)$$

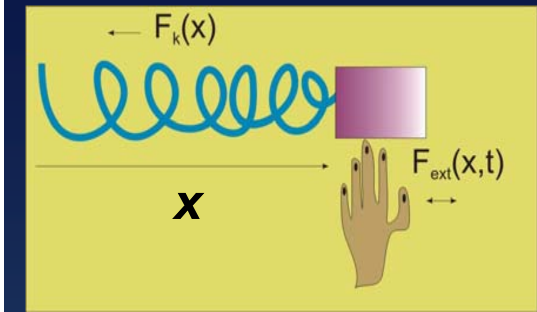
N **simultaneous 1st-order ODEs:**

$$\frac{dy^{(0)}(t)}{dt} = f^{(0)}[t, \mathbf{y}] \quad (19)$$

$$\frac{dy^{(1)}(t)}{dt} = f^{(1)}[t, \mathbf{y}] \quad (20)$$

$$\vdots \qquad \vdots$$

E.G.: Dynamical Form for 2nd-Order ODE (Problem!)



$$\frac{d^2x}{dt^2} = \frac{1}{m}F\left(t, \frac{dx}{dt}, x\right) \quad \overset{\Rightarrow}{\text{Standard form}} \quad \frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(t, \mathbf{y}) \tag{21}$$

- Rules: RHS no explicit derivatives, LHS only 1st derivatives
- Start: define position *x* = dependent variable
- Trick: define velocity = dependent variable

$$y^{(0)}(t) \overset{\text{def}}{=} x(t) \tag{22}$$

$$\mathbf{y}^{(1)}(t) \overset{\text{def}}{=} \frac{dx}{dt} \equiv \frac{dy^{(0)}}{dt} \tag{23}$$

We did it!

$$\frac{dy^{(0)}(t)}{dt} = y^{(1)}(t) \qquad\qquad f^{(0)} = y^{(1)}(t) \tag{24}$$

$$\frac{dy^{(1)}(t)}{dt} = \frac{1}{m}F(t, y^{(0)}, y^{(1)}) \quad \overset{\Rightarrow}{\text{standard form}} \quad f^{(1)} = F(t, y^{(0)}, y^{(1)}) \tag{25}$$