Shock Waves & Solitons

PDE Waves; Oft-Left-Out; CFD to Follow

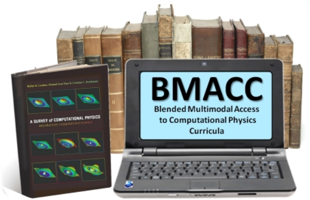
Rubin H Landau

Sally Haerer, Producer-Director

Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

with Support from the National Science Foundation

Course: **Computational Physics II**



Problem: Explain Russel's Observation

1834, J. Scott Russell, Edinburgh-Glasgow Canal

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon. . . ."

Problem: Explain Russel's Soliton Observation

J. Scott Russell, 1834, Edinburgh-Glasgow Canal

- We extend PDE Waves; You see String Waves 1st
- Extend: nonlinearities, dispersion, hydrodynamics
- Fluids, old but deep & challenging
- Equations: complicated, nonlinear, unstable, rare analytic
- Realistic BC $\neq$ intuitive (airplanes, autos)
- Solitons: computation essential, modern study

Theory: Advection = Continuity Equation

Simple Fluid Motion

- Continuity equation = conservation of mass

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = 0 \tag{1}$$

$$\mathbf{j}(\mathbf{x}, t) \stackrel{\text{def}}{=} \rho \mathbf{v} = \text{current} \tag{2}$$

- $\rho(\mathbf{x}, t)$ = mass density, $\mathbf{v}(\mathbf{x}, t)$ = fluid velocity
- $\vec{\nabla} \cdot \mathbf{j}$ = "Divergence" of current = spreading
- $\Delta\rho$: in + out current flow
- Advection Equation, 1-D flow, constant $v = c$:

$$\frac{\partial \rho(x, t)}{\partial t} + c\frac{\partial \rho(x, t)}{\partial x} = 0 \tag{3}$$

Solutions of Advection Equation

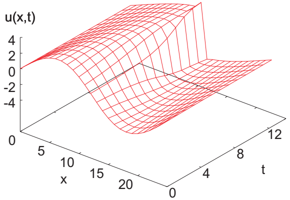
1st Derivative Wave Equation

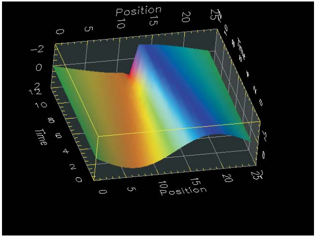
$$\frac{\partial \rho(x,t)}{\partial t} + c\frac{\partial \rho(x,t)}{\partial x} = 0$$

- "Advection" $\stackrel{\text{def}}{=}$ transport salt from thru water due to $\vec{v}$ field
- Solution: $u(x,t) = f(x-ct)$ = traveling wave
- Surfer rider on traveling wave crest
- Constant shape $\Rightarrow$

  $x - ct = \text{constant} \quad \Rightarrow \quad x = ct + C \quad \Rightarrow$ Surfer speed $= dx/dt = c$
- Can leapfrog, not for shocks

Extend Theory: Burgers' Equation





Wave Velocity $\propto$ Amplitude

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + \epsilon \frac{\partial (u^2/2)}{\partial x} = 0 \quad \text{(Conservative Form)} \qquad (2)$$

- Advection: all points @ c $\Rightarrow$ constant shape
- Burgers: larger amplitudes faster $\Rightarrow$ shock wave

Lax–Wendroff Algorithm for Burgers' Equation

Applet Going Beyond CD for Shocks

$$\frac{\partial u}{\partial t} + \epsilon \frac{\partial (u^2/2)}{\partial x} = 0 \quad \text{(Conservative Form)}$$

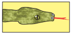
$$u(x, t + \Delta t) = u(x, t - \Delta t) - \beta \left[\frac{u^2(x + \Delta x, t) - u^2(x - \Delta x, t)}{2} \right]$$

$$\beta = \frac{\epsilon}{\Delta x / \Delta t} = \text{measure nonlinear} < 1 \quad \text{(stable)}$$

$$u(x, t + \Delta t) \simeq u(x, t) + \frac{\partial u}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} \Delta t^2$$

$$u_{i,j+1} = u_{i,j} - \frac{\beta}{4} \left(u^2_{i+1,j} - u^2_{i-1,j} \right) + \frac{\beta^2}{8} \Big[(u_{i+1,j} + u_{i,j}) \left(u^2_{i+1,j} - u^2_{i,j} \right)$$
$$- (u_{i,j} + u_{i-1,j}) \left(u^2_{i,j} - u^2_{i-1,j} \right) \Big]$$

Burger's Assessment



CODE

1. Solve Burgers' equation via leapfrog method
2. Study shock waves
3. Modify program to Lax–Wendroff method
4. Compare the leapfrog and Lax–Wendroff methods
5. Explore Δx and Δt
6. Check different β for stability
7. Separate numerical and physical instabilities

Dispersionless Propagation

Meaning of Dispersion?

- Dispersion $\not\Rightarrow$ E loss, Dispersion $\Rightarrow$ information loss
- Physical origin: propagate spatially regular medium
- Math origin: higher-order ∂_x
- $u(x,t) = e^{i(kx \mp \omega t)}$ = R/L "traveling" plane wave
- Dispersion Relation: sub into advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad (1)$$

$$\Rightarrow \quad \omega = \pm ck \quad \text{(dispersionless propagation)} \quad (2)$$

$$v_g = \frac{\partial \omega}{\partial k} = \quad \text{group velocity} = \pm c \quad \text{(linear)} \quad (3)$$

Including Dispersion (Wave Broadening)

Small-Dispersion Relation $w(k)$

- $\omega = ck$ = dispersionless

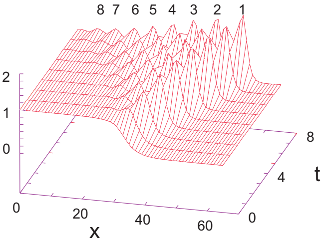
$$\omega \simeq ck - \beta k^3 \tag{1}$$

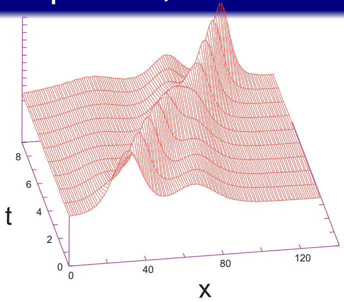
$$v_g = \frac{d\omega}{dk} \simeq c - 3\beta k^2 \tag{2}$$

- Even powers $\rightarrow$ R-L asymmetry in v_g
- Work back to wave equation, $k^3 \Rightarrow \partial_x^3$:

$$\frac{\partial u(x,t)}{\partial t} + c\frac{\partial u(x,t)}{\partial x} + \beta \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \tag{3}$$

Korteweg & deVries (KdeV) Equation, 1895





$$\frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t)\,\frac{\partial u(x,t)}{\partial x} + \mu\,\frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

- Nonlinear $\varepsilon u\,\partial u/\partial t$ $\rightarrow$sharpening $\rightarrow$ shock
- $\partial^3 u/\partial x^3$ $\rightarrow$ dispersion
- Stable: dispersion $\simeq$ shock; (parameters, IC)
- Rediscovered numerically Zabusky & Kruskal, 1965
- 8 Solitons, larger = faster, pass through each other!

Analytic Soliton Solution

Convert Nonlinear PDE to Linear ODE

- Guess traveling wave $\longrightarrow$ solvable ODE

$$0 = \frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t)\,\frac{\partial u(x,t)}{\partial x} + \mu\,\frac{\partial^3 u(x,t)}{\partial x^3} \quad (1)$$

$$u(x,t) = u(\xi = x - ct) \quad (2)$$

$$\Rightarrow \quad 0 = \frac{\partial u}{\partial \xi} + \epsilon\, u\,\frac{\partial u}{\partial \xi} + \mu\,\frac{d^3 u}{d\xi^3} \quad (3)$$

$$\Rightarrow \quad u(x,t) = \frac{-c}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}(x - ct - \xi_0)\right] \quad (4)$$

- $\operatorname{sech}^2 \Rightarrow$ solitary lump

Algorithm for KdeV Solitons

CD for ∂_t, ∂_x; 4 points ∂_x^3

$$u_{i,j+1} \simeq u_{i,j-1} - \frac{\epsilon}{3}\frac{\Delta t}{\Delta x}\left[u_{i+1,j} + u_{i,j} + u_{i-1,j}\right]\left[u_{i+1,j} - u_{i-1,j}\right]$$

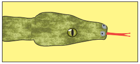
$$- \mu \frac{\Delta t}{(\Delta x)^3}\left[u_{i+2,j} + 2u_{i-1,j} - 2u_{i+1,j} - u_{i-2,j}\right]$$

- IC + FD to start (see text)
- Truncation error & stability:

$$\mathcal{E}(u) = \mathcal{O}[(\Delta t)^3] + \mathcal{O}[\Delta t(\Delta x)^2] \qquad ,$$

$$\frac{1}{(\Delta x/\Delta t)}\left[\epsilon|u| + 4\frac{\mu}{(\Delta x)^2}\right] \leq 1$$

Implementation: KdeV Solitons



CODE

- Bore $\longrightarrow$ solitons
- Solitons crossing
- Stability check
- Solitons in a box